APPLICATION OF THE GREEN'S FUNCTION METHOD IN FREE VIBRATION ANALYSIS OF NON-UNIFORM BEAMS

Stanisław Kukla, Izabela Zamojska

Institute of Mathematics and Computer Science, Czestochowa University of Technology

Abstract. In this paper an application of the Green's function method in frequency analysis of a beam with varying cross section is presented. The beam carries an arbitrary number of attached discrete systems. The exact solution of the problem concerns a beam with quadratically varrying cross-section area. Numerical examples show the influence of the selected parameters on free vibration frequencies of the considered system are presented.

Introduction

The free vibration problem for a uniform beams carrying various concentrated elements has been studied extensively by a lot of reaserchers. For example Wu and Chou [1] present vibration analysis of a uniform cantilever beam carrying any number of elastically mounted point masses by the application of the analytical-and-numerical-combined method. For a non-uniform beam, even without any attachements, exact solution of the vibration problem may be obtained only for some, fixed forms of varrying cross-section areas.

In literature there are presented the solutions of the vibration problems of non-uniform beams determined by using various methods. Abrate in his paper [2] to solve the problem uses the Rayleigh-Ritz approach. The exact solution for the beam with variable cross-section $A(\xi)$ and moment of inertia $I(\xi)$, with various boundary conditions and attached any number of spring-mass systems was presented in [3]. Chen and Wu use the numerical assembly method to perform the free vibration analysis of Bernoulli-Euler non-uniform beam carrying any number of concentrated attachements.

This paper presents the exact solution of the vibration problem of the non-uniform beam which is obtained by using the Green's function method. The problem formulation constitutes the Bernoulli-Euler differential equation and the boundary conditions corresponding to the cantilever beam. It is assumed that on the beam an arbitrary number of discrete elements (masses or translational springs) are attached. The neccesary Green's function corresponding to the considered beam is determined. The solution of the problem concerns the beams with quadratically varrying cross-section areas. Using the obtained frequency equation of the considered vibrational system, the numerical analysis is performed.

1. Formulation and solution of the problem

Consider a non-uniform beam of length L with n springs and/or masses attached at points x_i, i = 1,2,…,n, of the beam (Fig. 1). It is assumed that b_0, h_0 are the widht and height of the cross-section at x = 0, b_L, h_L are width and height at x = L, respectively. According to the Euler-Bernoulli theory, the equation of motion for the considering beam is

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]+\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}=f(x,t) \quad (1)$$

where A(x) is the cross-section area at the position x, I(x) is the moment of inertia of A(x), E denotes Young's modulus, ρ is the mass density of the beam material, w(x,t) is the transverse deflection at position x and time t. The function w(x,t) satysfies homogeneous boundary conditions which may be symbolically written in the following form:

$$\mathbf{B}_0\left[w(x,t)\right]\Big|_{x=0}=0\,,\ \mathbf{B}_1\left[w(x,t)\right]\Big|_{x=L}=0 \quad (2)$$

where $\mathbf{B_0}$ and $\mathbf{B_1}$ are two dimensional "vectors", the components of which are linear, spatial differential operators.

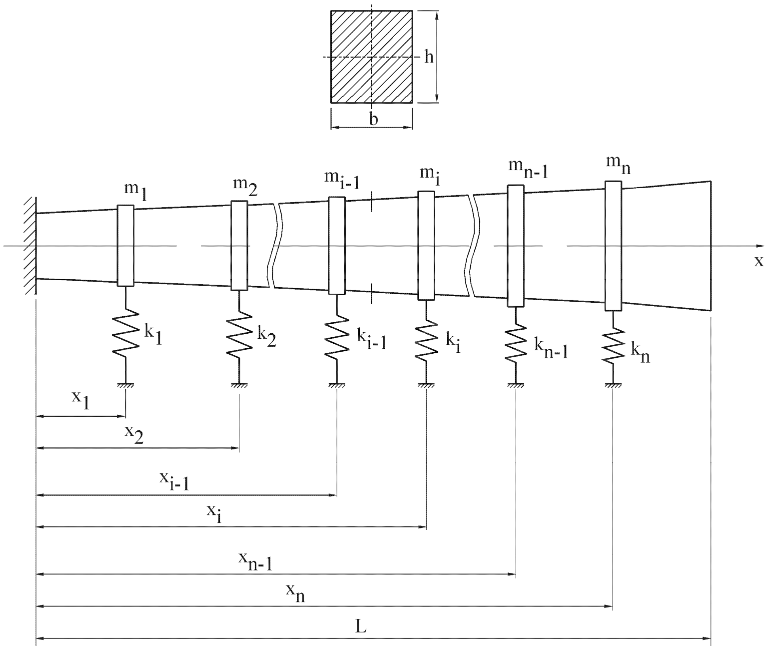

Fig. 1. A sketch of a considering non-uniform beam with an additional spring-mass discrete elements

The form of the function f(x,t) depends on the nature of attached discrete systems. In the considered problem, the function is assumed in the following form:

$$f(x,t) = -\sum_{j=1}^{n}\left[m_j \frac{\partial^2 w(x,t)}{\partial t^2} + k_j w(x,t)\right]\delta(x - x_j) \tag{3}$$

where δ( ) is the Dirac delta function, m_i and k_i denote masses and spring constants characterising the mounted discrete elements, respectively.

Free vibrations of the beam are harmonic, i.e. the function w(x,t) takes the form

$$w(x,t) = \bar{W}(x)e^{i\omega t} \tag{4}$$

where $i = \sqrt{-1}$, ω is the natural frequency of the beam and $\bar{W}(x)$ is the amplitude of deflection. Further, it is assumed that $b_L/b_0 = h_L/h_0 = \alpha$ and

$$A(x) = A_0\left(\frac{\alpha-1}{L}x+1\right)^2, \quad I(x) = I_0\left(\frac{\alpha-1}{L}x+1\right)^4 \tag{5}$$

where $\alpha \neq 1$, $A_0 = A(0)$, $I_0 = I(0)$ in equation (1).

The substitution of the equations (4) and (5) into the equation (1) yields

$$\frac{d^2}{dx^2}\left[EI_0\left(\frac{\alpha-1}{L}x+1\right)^4 \frac{d^2\bar{W}(x)}{dx^2}\right] - \omega^2\rho A_0\left(\frac{\alpha-1}{L}x+1\right)^2 \bar{W}(x) =$$

$$= \sum_{j=1}^{n}\left[m_j\omega^2 - k_j\right]\bar{W}(x)\delta(x - x_j) \tag{6}$$

Next, the non-dimensional coefficient

$$\xi = \frac{\alpha-1}{L}x+1, \quad \xi \in \langle 1,\alpha \rangle \tag{7}$$

is introduced into the differential equation (6) and the boundary conditions (2). The differential equation and the boundary conditions take the following form:

$$\frac{d^2}{d\xi^2}\left[\xi^4 \frac{d^2W(\xi)}{d\xi^2}\right] - \left(\frac{\beta}{2}\right)^4 \xi^2 W(\xi) = \sum_{j=1}^{n}\mu_j\left[\omega^2 - \Lambda_j^2\right]W(\xi)\delta(\xi - \eta_j) \tag{8}$$

$$\bar{\mathbf{B}}_0\left[W(\xi)\right]\Big|_{\xi=1} = 0\,, \ \bar{\mathbf{B}}_1\left[W(\xi)\right]\Big|_{\xi=\alpha} = 0 \tag{9}$$

where $\Omega^4 = \omega^2 \rho A_0 / EI_0$, $\beta = 2L\Omega/(\alpha-1)$, $\mu_i = m_i L^3 / EI_0 (\alpha-1)^3$, $\Lambda_j = \sqrt{k_j/m_j}$ and $\eta_i = 1 + (\alpha-1) x_i / L$ for j = 1,2,...,n.

The solution of the problem (8)-(9) is obtained by using the Green's function method. Assuming, that the Green's function is known, the solution of the considered boundary problem can be expressed as the following sum [4]:

$$W(\xi) = \sum_{j=1}^{n} \mu_j \left[\omega^2 - \Lambda_j^2\right] W(\eta_j) \delta(\xi - \eta_j) \tag{10}$$

Substituting $\xi = \eta_j$ j = 1,2,...,n, successively into equation (10), a set of n equations with unknown $W(\eta_j)$ (j = 1,2,…,n) one obtains. The determinant of the coefficient matrix of the equations system must disappear to exist non-trivial solution of the considered boundary problem. It yields the frequency equation

$$\det \mathbf{A} = 0 \tag{11}$$

where $\mathbf{A} = \left[a_{ij}\right]_{1\le i,j\le n}$, $a_{ij} = \mu_j \left[\omega^2 - \Lambda_j^2\right] G(\eta_i, \eta_j) + \delta_{ij}$ and δ_{ij} is the Kronecker delta. The equation (11), with unknown ω, is then solved numerically.

2. Green's function

The Green's function of the differentional operator

$$\mathbf{L} = \frac{d^2}{d\xi^2}\left[\xi^4 \frac{d^2}{d\xi^2}\right] - \left(\frac{\beta}{2}\right)^4 \xi^2 \tag{12}$$

satisfies the equation

$$\mathbf{L}\left[G(\xi,\eta)\right] = \delta(\xi - \eta) \tag{13}$$

and the following boundary conditions

$$\bar{\mathbf{B}}_0\left[G(\xi,\eta)\right]\Big|_{\xi=1} = 0,\ \bar{\mathbf{B}}_1\left[G(\xi,\eta)\right]\Big|_{\xi=\alpha} = 0 \tag{14}$$

The function G may be written as a sum [4] of function G_0 which is the general solution of the homogeneous equation $\mathbf{L}\left[G_0(\xi,\eta)\right] = 0$ and G_S which is the particular solution of the equation (13). The particular solution G_S can be written in the form: $G_S(\xi,\eta) = G_1(\xi,\eta)H(\xi-\eta)$ where H( ) is the Heaviside function. The function G_1 fulfill the homogeneous equation [4]

$$\mathbf{L}\left[G_1\right] = 0 \tag{15}$$

and the conditions below:

$$G_1(\xi,\eta)\big|_{\xi=\eta} = \left.\frac{dG_1(\xi,\eta)}{d\xi}\right|_{\xi=\eta} = \left.\frac{d^2G_1(\xi,\eta)}{d\xi^2}\right|_{\xi=\eta} = 0, \quad \left.\frac{d^3G_1(\xi,\eta)}{d\xi^3}\right|_{\xi=\eta} = \frac{1}{\eta^4} \tag{16}$$

The general solution of (15) is expressed by well-known Bessel functions [3, 5]:

$$G_1(\xi,\eta) = \xi^{-1}\left[c_1 J_2\left(\beta\sqrt{\xi}\right) + c_2 Y_2\left(\beta\sqrt{\xi}\right) + c_3 I_2\left(\beta\sqrt{\xi}\right) + c_4 K_2\left(\beta\sqrt{\xi}\right)\right] \tag{17}$$

Taking into account (17) in equations (16), one obtains a set of four equations with unknown c_1, c_2, c_3, c_4. After the solution of the equation system, the function $G_1(\xi,\eta)$ can be written in the form:

$$\begin{aligned} G_1(\xi,\eta) = 4\left(\beta^2\eta\xi\right)^{-1}\Big[& I_2\left(\beta\sqrt{\xi}\right)K_2\left(\beta\sqrt{\eta}\right) - I_2\left(\beta\sqrt{\eta}\right)K_2\left(\beta\sqrt{\xi}\right) + \\ & + \frac{\pi}{2}\left(J_2\left(\beta\sqrt{\xi}\right)Y_2\left(\beta\sqrt{\eta}\right) - J_2\left(\beta\sqrt{\eta}\right)Y_2\left(\beta\sqrt{\xi}\right)\right)\Big] \end{aligned} \tag{18}$$

Finally, the Green's function of the differential problem (13)-(14) takes the form:

$$\begin{aligned} G(\xi,\eta) = {} & \xi^{-1}\left[C_1 J_2\left(\beta\sqrt{\xi}\right) + C_2 Y_2\left(\beta\sqrt{\xi}\right) + C_3 I_2\left(\beta\sqrt{\xi}\right) + C_4 K_2\left(\beta\sqrt{\xi}\right)\right] + \\ & + G_1(\xi,\eta) H(\xi-\eta) \end{aligned} \tag{19}$$

Unknown coefficients C_i, i = 1,2,3,4, occurring in the equation (19), are determined on the basis of the boundary conditions (14).

For example, the conditions (14) for a cantilever beam (Fig. 1) are as follows:

$$G(\xi,\eta)\big|_{\xi=1} = \left.\frac{dG(\xi,\eta)}{d\xi}\right|_{\xi=1} = 0, \quad \left.\frac{d^2G(\xi,\eta)}{d\xi^2}\right|_{\xi=\alpha} = \left.\frac{d^3G(\xi,\eta)}{d\xi^3}\right|_{\xi=\alpha} = 0 \tag{20}$$

The Green's function $G(\xi,\eta)$ of the differential operator **L** is defined by the equations (18) and (19) where coefficients C_i for i = 1,2,3,4 are as follows:

$$C_1 = \frac{-2}{\beta^2\eta}\cdot\frac{f(\alpha,\eta)b(\alpha) + e(\alpha,\eta)d(\alpha)}{w(\alpha)}, \quad C_2 = \frac{2}{\beta^2\eta}\cdot\frac{f(\alpha,\eta)a(\alpha) + e(\alpha,\eta)c(\alpha)}{w(\alpha)},$$

$$C_3 = \phi_1 C_1 + \phi_2 C_2, \quad C_4 = \phi_3 C_1 + \phi_4 C_2 \tag{21}$$

The functions $a(\alpha)$, $b(\alpha)$, $c(\alpha)$, $d(\alpha)$, $e(\alpha,\eta)$, $f(\alpha,\eta)$ are further presented in Appendix.

3. Numerical examples

Assuming just one discrete spring-mass element, mounted at free end of the beam, the frequency equation (11) is obtained in the form

$$1+\mu_1\left(\Lambda_1^2-\omega^2\right)G(\eta,\eta)=0 \tag{22}$$

where $\eta = \alpha$. The Green's function in (22) takes form (18), where G1 is presented in (19) and coefficients C_i (i = 1,2,3,4) are defined in (21).

In the numerical examples the following values of parameters characterizing physical and geometrical properties of the beam are assumed: L = 1.0 m, $E = 2.069 \cdot 10^{11}$ N/m^2, $b_0 = 0.02$ m, $h_0 = 0.04$ m, $\rho = 7950$ kg/m^3, $m_1 = m^* \cdot \rho A_0 L\left[\frac{1}{3}(\alpha-1)^2+\alpha\right]$, $k_1 = k^* \cdot EI_0/L^3$, A(x) and I(x) are defined by (4).

Constant value $m^* = 0.2$ means that mass point is one fifth of the total mass of the beam, $k^* = 3.0$ represents one third of the spring constant of the considered beam.

Table 1

Nondimensional frequency parameter values Ω_i for the first four modes of vibration for a clamped-free beam with a cross-section area parameter α

α	Ω_1	Ω_2	Ω_3	Ω_4
3	1.43819	5.19175	9.89457	14.22343
2.75	1.47253	5.10765	9.63322	13.83672
2.5	1.51106	5.01820	9.36038	13.43245
2.25	1.55474	4.92243	9.07420	13.00786
2	1.60486	4.81902	8.77228	12.55942
1.75	1.66318	4.70614	8.45138	12.08244
1.5	1.73216	4.58113	8.10705	11.57054
1.25	1.81537	4.43992	7.73288	11.01473

Table 1 shows the first four non-dimensional frequency parameters Ω_i (i = 1,2,3,4) of the beam as a function of parameter $\alpha = b_L/b_0 = h_L/h_0$. It may be observed that increase of the value of α (parameter characterizing the non-uniformity of the beam) causes decrease of the first frequency parameter Ω_1 and increase of the Ω_i for i = 2,3,4.

In the case of the cantilever beam carrying one discrete element at $\eta_1 \in \langle 1,\alpha\rangle$, the frequency equation is obtained by assuming $\eta = \eta_1$ in equation (22). Figure 2 presents frequency parameter values Ω_i for the first four modes of vibration as functions of η_1 for the considered beam with the various cross-section $A(\xi) = \xi^2$

and the moment of inertia $I(\xi) = \xi^4$ where ξ is given by eg. (7) and $\alpha = 2.0$ (Fig. 2a), $\alpha = 3.0$ (Fig. 2b). Figure 2 shows that change of the location of the spring-mass system on the beam carries weight on the higher frequency values.

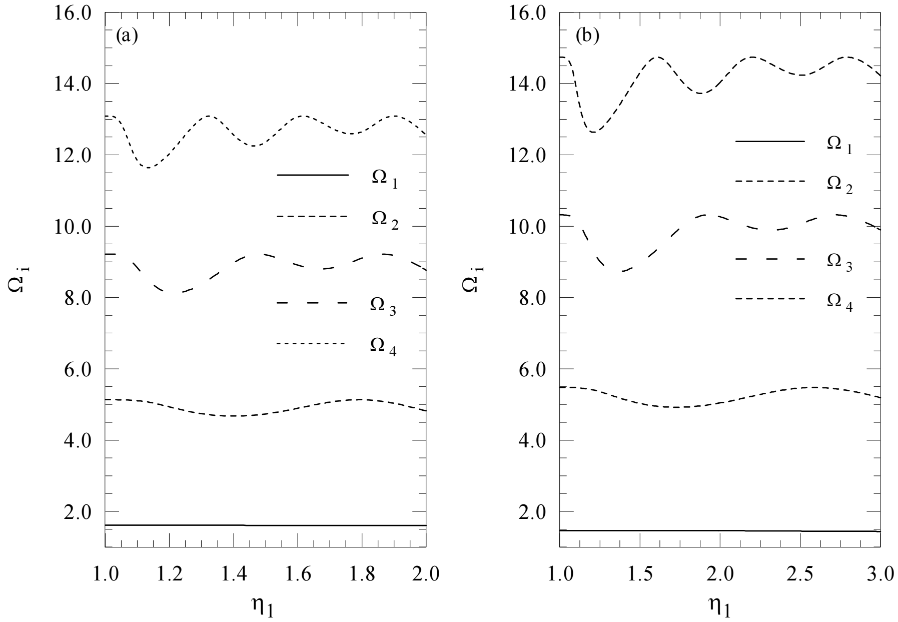

Fig. 2. Frequency parameter values Ω_i for the first four modes of vibration as a function of η_1 : a) $\alpha = 2.0$, b) $\alpha = 3.0$

Conclusions

The free vibration problem of a non-uniform beam carrying any number of discrete elements is the subject of this paper. The exact solution of the problem has been obtained by the application of the Green's function method. The numerical examples have shown the effect of non-uniformity of the cantilever beam on the eigenfrequencies of the system. Moreover, the numerical investigations present the eigenfrequencies of the system as functions of the location of one spring-mass element attached to the cantilever beam with variable cross-section area. Although the solution of the problem concerns the cantilever beam, the presented method can be used for boundary conditions corresponding to other beams.

References

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Appendix

The functions occuring in definition of the coefficients C_i (equation (21)) are given as follows:

$$\phi_1 = -\beta\left[J_2(\beta)K_1(\beta) + J_1(\beta)K_2(\beta)\right], \quad \phi_2 = -\beta\left[K_2(\beta)Y_1(\beta) + K_1(\beta)Y_2(\beta)\right],$$

$$\phi_3 = \beta\left[I_2(\beta)J_1(\beta) - I_1(\beta)J_2(\beta)\right], \quad \phi_4 = \beta\left[I_2(\beta)Y_1(\beta) - I_1(\beta)Y_2(\beta)\right]$$

$$a(\alpha) = J_4(\theta) + I_4(\theta)\phi_1 + K_4(\theta)\phi_3, \quad b(\alpha) = Y_4(\theta) + I_4(\theta)\phi_2 + K_4(\theta)\phi_4,$$

$$c(\alpha) = J_5(\theta) - I_5(\theta)\phi_1 + K_5(\theta)\phi_3, \quad d(\alpha) = Y_5(\theta) - I_5(\theta)\phi_2 + K_5(\theta)\phi_4,$$

$$w(\alpha) = a(\alpha)d(\alpha) - c(\alpha)b(\alpha),$$

$$e(\alpha,\eta) = 2\left[I_4(\theta)K_2(\sigma) - I_2(\sigma)K_4(\theta)\right] + \pi\left[J_4(\theta)Y_2(\sigma) - J_2(\sigma)Y_4(\theta)\right],$$

$$f(\alpha,\eta) = 2\left[I_5(\theta)K_2(\sigma) + I_2(\sigma)K_5(\theta)\right] - \pi\left[J_5(\theta)Y_2(\sigma) - J_2(\sigma)Y_5(\theta)\right]$$

where: $\theta = \beta\sqrt{\alpha}$, $\sigma = \beta\sqrt{\eta}$.