# DUAL RECIPROCITY BOUNDARY METHOD FOR THE POISSON EQUATION

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**Abstract.** The standard boundary element method for Poisson equation requires the discretization of boundary and interior of the domain considered. In this paper the variant called dual reciprocity boundary element method is presented. On the stage of numerical computations this approach allows to avoid the discretization of the interior of domain. In the final part of the paper the example of computations and comparison of results obtained using the BEM and DRBEM are shown.

## 1. Governing equations

We consider the Poisson equation

$$(x,y) \in \Omega: \quad \lambda \nabla^2 T(x,y) + Q(x,y) = 0 \tag{1}$$

where $\lambda[\mathrm{W/mK}]$ is the thermal conductivity, $T$ is the temperature, $x, y$ are the geometrical co-ordinates, $Q(x,y)$ is the source function. The equation (1) is supplemented by the boundary conditions:

$$\begin{aligned} &(x,y) \in \Gamma_1 : T(x,y) = T_b \\ &(x,y) \in \Gamma_2 : q(x,y) = -\lambda \mathbf{n} \cdot \nabla T(x,y) = q_b \end{aligned} \tag{2}$$

where $T_b$ is known boundary temperature, $\mathbf{n}$ is the normal outward vector at the boundary point $(x,y)$, $q_b$ is given boundary heat flux.

## 2. Boundary element method

The boundary integral equation for equation (1) is following [1, 2]:

$$\begin{aligned} &(\xi,\eta) \in \Gamma: \quad B(\xi,\eta)\,T(\xi,\eta) + \int_\Gamma q(x,y)\,T^*(\xi,\eta,x,y)\,\mathrm{d}\Gamma = \\ &\int_\Gamma T(x,y)\,q^*(\xi,\eta,x,y)\,\mathrm{d}\Gamma + \iint_\Omega Q(x,y)\,T^*(\xi,\eta,x,y)\,\mathrm{d}\Omega \end{aligned} \tag{3}$$

where $B(\xi,\eta)$ is the coefficient connected with the local shape of the boundary, $T^*(\xi,\eta,x,y)$ is the fundamental solution, $(\xi,\eta)$ is the observation point, while

$$q^*(\xi,\eta,x,y) = -\lambda\,\mathbf{n}\cdot\nabla T^*(\xi,\eta,x,y) \tag{4}$$

and

$$q(x,y) = -\lambda\,\mathbf{n}\cdot\nabla T(x,y) \tag{5}$$

Fundamental solution has the following form

$$T^*(\xi,\eta,x,y) = \frac{1}{2\pi\lambda}\ln\frac{1}{r} \tag{6}$$

where $r$ is the distance between the points $(\xi,\eta)$ and $(x,y)$

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \tag{7}$$

Heat flux resulting from the fundamental solution can be calculated analytically

$$q^*(\xi,\eta,x,y) = \frac{d}{2\pi r^2} \tag{8}$$

where

$$d = (x-\xi)\cos\alpha + (y-\eta)\cos\beta \tag{9}$$

while $\cos\alpha$, $\cos\beta$ are the directional cosines of the boundary normal vector $\mathbf{n}$.

## 3. Dual reciprocity boundary method

The solution of Poisson equation (1) can be written as a sum

$$T(x,y) = \hat{T}(x,y) + U(x,y) \tag{10}$$

where the first function is the solution of Laplace's equation

$$(x,y)\in\Omega: \quad \lambda\nabla^2\hat{T}(x,y) = 0 \tag{11}$$

while $U(x,y)$ is the particular solution

$$(x,y)\in\Omega: \quad \lambda\nabla^2 U(x,y) = -Q(x,y) \tag{12}$$

It is generally difficult to find a solution $U(x,y)$, so in the dual reciprocity method the following approximation for $Q(x,y)$ is proposed [3]

$$Q(x,y) \approx \sum_{k=1}^{N+L} a_k f_k(x,y) \tag{13}$$

where $a_k$ are unknown coefficients and $f_k(x,y)$ are approximating functions fulfilling the equations

$$-\lambda \nabla^2 U_k(x,y) = f_k(x,y) \tag{14}$$

In equation (13) $N+L$ corresponds to the total number of nodes, where $N$ is the number of boundary nodes and $L$ is the number of internal nodes.
Putting (14) into (13) one has

$$Q(x,y) = -\lambda \sum_{k=1}^{N+L} a_k \nabla^2 U_k(x,y) \tag{15}$$

We consider the last integral in equation (3)

$$\begin{gathered} D = \iint_\Omega Q(x,y)\, T^*(\xi,\eta,x,y)\, \mathrm{d}\Omega = \\ -\sum_{k=1}^{N+L} a_k \iint_\Omega \left[\lambda \nabla^2 U_k(x,y)\right] T^*(\xi,\eta,x,y) \mathrm{d}\Omega \end{gathered} \tag{16}$$

Using the second Green formula one obtains

$$\begin{gathered} D = -\sum_{k=1}^{N+L} a_k \iint_\Omega \left[\lambda \nabla^2 T^*(\xi,\eta,x,y)\right] U_k(x,y)\, \mathrm{d}\Omega - \\ \sum_{k=1}^{N+L} a_k \int_\Gamma \left[\lambda T^*(\xi,\eta,x,y) \mathbf{n} \cdot \nabla U_k(x,y) - \lambda U_k(x,y) \mathbf{n} \cdot \nabla T^*(\xi,\eta,x,y)\right] \mathrm{d}\Gamma \end{gathered} \tag{17}$$

or

$$\begin{gathered} D = \sum_{k=1}^{N+L} a_k \Big[ B(\xi,\eta)\, U_k(\xi,\eta) + \\ \int_\Gamma T^*(\xi,\eta,x,y)\, W_k(x,y)\, \mathrm{d}\Gamma - \int_\Gamma U_k(x,y)\, q^*(\xi,\eta,x,y)\, \mathrm{d}\Gamma \Big] \end{gathered} \tag{18}$$

where

$$W_k(x,y) = -\lambda \mathbf{n} \cdot \nabla U_k(x,y) \tag{19}$$

So, the equation (3) takes the following form

$$
\begin{gathered}
B(\xi,\eta)\,T(\xi,\eta) + \int_{\Gamma} q(x,y)\,T^{*}(\xi,\eta,x,y)\,\mathrm{d}\Gamma = \\
\int_{\Gamma} T(x,y)\,q^{*}(\xi,\eta,x,y)\,\mathrm{d}\Gamma + \sum_{k=1}^{N+L} a_k \Big[ B(\xi,\eta)\,U_k(\xi,\eta) + \\
\int_{\Gamma} T^{*}(\xi,\eta,x,y)\,W_k(x,y)\,\mathrm{d}\Gamma - \int_{\Gamma} q^{*}(\xi,\eta,x,y)\,U_k(x,y)\,\mathrm{d}\Gamma \Big]
\end{gathered}
\tag{20}
$$

In numerical realization of DRBEM the boundary is divided into $N$ constant boundary elements and $L$ internal nodes are distinguished. The integral appearing in equation (20) are substituted by the sum of integrals over the boundary elements and then

$$
\frac{1}{2}T_i + \sum_{j=1}^{N} G_{ij}\,q_j = \sum_{j=1}^{N} \hat{H}_{ij}\,T_j + \sum_{k=1}^{N+L} a_k \left( \frac{1}{2}U_{ik} + \sum_{j=1}^{N} G_{ij}\,W_{jk} - \sum_{j=1}^{N} \hat{H}_{ij}\,U_{jk} \right) \tag{21}
$$

or

$$
\sum_{j=1}^{N} G_{ij}\,q_j = \sum_{j=1}^{N} H_{ij}\,T_j + \sum_{k=1}^{N+L} a_k \left( \sum_{j=1}^{N} G_{ij}\,W_{jk} - \sum_{j=1}^{N} H_{ij}\,U_{jk} \right) \tag{22}
$$

where

$$
G_{ij} = \int_{\Gamma_j} T^{*}(\xi_i,\eta_i,x,y)\,\mathrm{d}\Gamma_j = \frac{1}{2\pi\lambda} \int_{\Gamma_j} \ln\left(\frac{1}{r_{ij}}\right) \mathrm{d}\Gamma_j \tag{23}
$$

and

$$
H_{ij} = \begin{cases} \displaystyle\int_{\Gamma_j} q^{*}(\xi_i,\eta_i,x,y)\,\mathrm{d}\Gamma_j = \frac{1}{2\pi} \int_{\Gamma_j} \frac{d_{ij}}{r_{ij}^2}\,\mathrm{d}\Gamma_j\,, & i \neq j \\ -1/2\,, & i = j \end{cases} \tag{24}
$$

We define

$$
U_{jk} = \frac{r_{jk}^2}{4} + \frac{r_{jk}^3}{9} \tag{25}
$$

where

$$
r_{jk}^2 = \left(x_k - x_j\right)^2 + \left(y_k - y_j\right)^2 \tag{26}
$$

Using the formula (19) we obtain

$$W_{jk} = -\lambda\, d_{jk} \left( \frac{1}{2} + \frac{1}{3} r_{jk} \right) \tag{27}$$

where

$$d_{jk} = \left( x_k - x_j \right) \cos\alpha_k + \left( y_k - y_j \right) \cos\beta_k \tag{28}$$

Because

$$\nabla^2 U_{sk} = 1 + r_{sk} \tag{29}$$

so on the basis of equation (14) one has

$$f_{sk} = -\lambda \left( 1 + r_{sk} \right) \tag{30}$$

The equation (13) can be expressed as follows

$$Q_s = -\lambda \sum_{k=1}^{N+L} a_k \left( 1 + r_{sk} \right) \quad , \qquad s = 1, 2, ..., N + L \tag{31}$$

The system of equations (31) can be written in the matrix form

$$\begin{bmatrix} Q_1 \\ Q_2 \\ \cdots \\ Q_{N+L} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1\,N+L} \\ f_{21} & f_{22} & \cdots & f_{2\,N+L} \\ \cdots & \cdots & \cdots & \cdots \\ f_{N+L,\,1} & f_{N+L,\,2} & \cdots & f_{N+L,\,N+L} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_{N+L} \end{bmatrix} \tag{32}$$

This system allows to calculate the coefficients $a_k$, $k = 1, 2, ..., N + L$. The equations (22) can be also written in the matrix form

$$\mathbf{G\,q} = \mathbf{HT} + \left( \mathbf{G\,W} - \mathbf{HU} \right) \mathbf{a} \tag{33}$$

where

$$\mathbf{G} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1\,N} \\ G_{21} & G_{22} & \cdots & G_{2\,N} \\ \cdots & \cdots & \cdots & \cdots \\ G_{N,\,1} & G_{N,\,2} & \cdots & G_{N,\,N} \end{bmatrix} \tag{34}$$

$$\mathbf{H} = \begin{bmatrix} H_{11} & H_{12} & \ldots & H_{1\,N} \\ H_{21} & H_{22} & \ldots & H_{2\,N} \\ \ldots & \ldots & \ldots & \ldots \\ H_{N,1} & H_{N,2} & \ldots & H_{N,N} \end{bmatrix} \tag{35}$$

and

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & \ldots & U_{1\,N+L} \\ U_{21} & U_{22} & \ldots & U_{2\,N+L} \\ \ldots & \ldots & \ldots & \ldots \\ U_{N,1} & U_{N,2} & \ldots & U_{N,N+L} \end{bmatrix} \tag{36}$$

while

$$\mathbf{W} = \begin{bmatrix} W_{11} & W_{12} & \ldots & W_{1\,N+L} \\ W_{21} & W_{22} & \ldots & W_{2\,N+L} \\ \ldots & \ldots & \ldots & \ldots \\ W_{N,1} & W_{N,2} & \ldots & W_{N,N+L} \end{bmatrix} \tag{37}$$

After solving the system of equations (33), the temperatures and heat fluxes at boundary nodes are known. Next, the temperatures at the internal nodes are calculated using the formula (c.f. equation (21))

$$T_i = \sum_{j=1}^{N} H_{ij} T_j - \sum_{j=1}^{N} G_{ij} q_j + \sum_{k=1}^{N+L} a_k \left( U_{ik} + \sum_{j=1}^{N} G_{ij} W_{jk} - \sum_{j=1}^{N} \hat{H}_{ij} U_{jk} \right) \tag{38}$$

## 4. Example of computations

The square of dimensions 0.03×0.03 m has been considered. Thermal conductivity equals $\lambda = 1$ W/(mK). On the left part of the boundary the Neumann condition $q_b = -10^4$ W/m$^2$ has been assumed, on the remaining parts of the boundary the Dirichlet condition $T_b = 100^\circ\mathrm{C}$ has been accepted. The boundary has been divided into 20 constant boundary elements, 25 internal nodes are distinguished. In order to compare the results obtained using DRBEM with the results obtained by BEM, in the second case the interior has been divided into 25 constant internal cells (Fig. 2).

The calculations have been done for three different source functions, this means:

1. $Q(x,y) = 10^7\left(x^2 + y^2\right)$

2. $Q(x,y) = 10^7\left(x^3 + y^3\right)$

3. $Q(x,y) = 10^7 x^2 + 5\cdot 10^7 y^2$.

In the Table 1 the results of computations are shown.

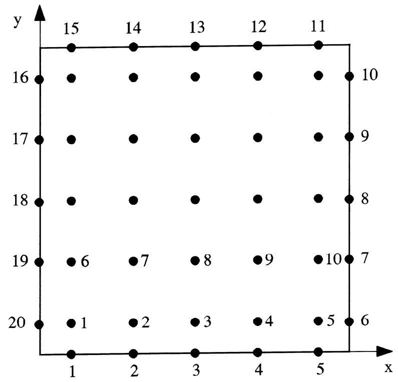


Fig. 1. Discretization and internal nodes (DRBEM)

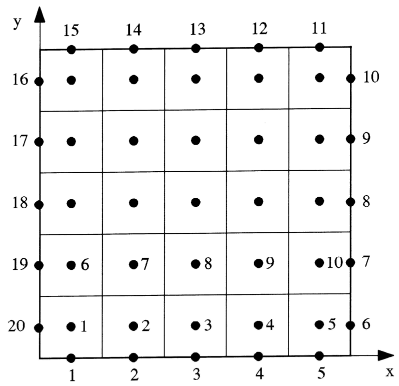


Fig. 2. Discretization (BEM)

Table 1

**Comparison of the BEM and DRBEM**

| Internal node | Variant 1 | | Variant 2 | | Variant 3 | |
|---|---|---|---|---|---|---|
| | BEM | DRBEM | BEM | DRBEM | BEM | DRMEB |
| 1 | 430 | 429 | 433 | 433 | 428 | 422 |
| 2 | 245 | 247 | 248 | 248 | 243 | 245 |
| 3 | 171 | 174 | 174 | 175 | 170 | 173 |
| 4 | 132 | 135 | 135 | 135 | 130 | 134 |
| 5 | 107 | 110 | 110 | 110 | 105 | 110 |

## References


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