SECOND ORDER SENSITIVITY ANALYSIS OF HEAT CONDUCTION PROBLEMS

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Abstract. In the paper the selected problems of sensitivity analysis application in the numerical modeling of heat conduction processes are discussed. The model of heat transfer bases on the Fourier equation supplemented by the geometrical, physical, boundary and initial conditions. In the first part of the paper the problems for which the second order sensitivity $V(x, t) = 0$, while in the next part the problems for which the second order sensitivity can be taken into account are presented, at the same time the direct approach of sensitivity analysis is used. On the stage of numerical computations the finite difference method [1] is applied.

1. Governing equations

The transient temperature field in the solid domain is determined by the following energy equation

$$x \in \Omega \quad : \quad c\frac{\partial T(x,t)}{\partial t} = \lambda \nabla^2 T(x,t) + Q \tag{1}$$

where $T(x, t)$ is the temperature, c is the volumetric specific heat, λ is the thermal conductivity, Q is the capacity of internal heat sources, x, t denote the spatial co-ordinates and time. Let us assume that on the external surface of Ω the condition is general form, this means

$$x \in \Gamma_0 \quad : \quad \Phi\left[\frac{\partial T(x,t)}{\partial n}, T(x,t)\right] = 0 \tag{2}$$

is given, $\partial(\cdot)/\partial n$ denotes the normal derivative.
For $t = 0$ the initial temperature is known

$$t = 0 \quad : \quad T(x,0) = T_0(x) \tag{3}$$

2. Sensitivity model independent of the basic model

At first the sensitivity of the problem discussed with respect to the boundary temperature T_b (Dirichlet problem) will be analyzed. According to the rules of direct approach [2-5] the Fourier equation and the boundary - initial conditions must be differentiated with respect to T_b. So using the Schwarz theorem one obtains

$$\begin{cases} x \in \Omega & : \quad c\dfrac{\partial U(x,t)}{\partial t} = \lambda \nabla^2 U(x,t) \\ x \in \Gamma_0 & : \quad U(x,t) = 1 \\ t = 0 & : \quad U(x,0) = 0 \end{cases} \tag{4}$$

where $U = \partial T/\partial T_b$. One can notice that in the case discussed the sensitivity model (4) is independent of the basic one (1)-(3). In such a case the sensitivity problem can be solved separately.

The second order sensitivity is defined as $V = \partial U/\partial p = \partial^2 T/\partial p^2$, where p is the parameter considered. In the case of problem (4) we obtain the following model of second order sensitivity

$$\begin{cases} x \in \Omega & : \quad c\dfrac{\partial V(x,t)}{\partial t} = \lambda \nabla^2 V(x,t) \\ x \in \Gamma_0 & : \quad V(x,t) = 0 \\ t = 0 & : \quad V(x,0) = 0 \end{cases} \tag{5}$$

where $V = \partial U/\partial T_b$ (or $\partial^2 T/\partial T_b^2$). Taking into account the form of boundary and initial conditions the solution of problem (5) is very simple, namely $V(x, t) = 0$.

Let us assume that the basic solution of problem (1)-(3) is known and we want to 'rebuilt' this result on the solution concerning the new boundary temperature T_b. Then using the Taylor formula we have

$$T(x,t,T_b \pm \Delta T_b) = T(x,t,T_b) \pm U(x,t,T_b)\Delta T_b \tag{6}$$

and it is the exact formula. So the precise 'rebuilding' of temperature field found for boundary temperature T_b on the solution for $T_b \pm \Delta T_b$ is possible for optional value of ΔT_b.

The considerations presented above can be illustrated by the following example. We consider the 1D cylindrical layer (external radius 0.01 m, internal radius 0.03 m). Thermophysical parameters of domain Ω equal λ = 1.3 W/mK, c = 1.6 MJ/m^{3}K, Q = 0, initial temperature T_0 = 20°C, external boundary temperature T_b = 100°C and next T_b = 150°C.

On the internal surface the no-flux condition is assumed. So, the following boundary-initial problem is considered

$$\begin{cases} x \in \Omega & : \quad x\dfrac{\partial T(x,t)}{\partial t} = \dfrac{\lambda}{x}\dfrac{\partial}{\partial x}\left[x\dfrac{\partial T(x,t)}{\partial x}\right] \\ x = R_2 & : \quad T(x,t) = T_b \\ x = R_1 & : \quad -\lambda\dfrac{\partial T(x,t)}{\partial x} = 0 \\ t = 0 & : \quad T(x,0) = 0 \end{cases} \tag{7}$$

and the sensitivity model wit respect to T_b is of the form

$$\begin{cases} x \in \Omega & : \quad x\dfrac{\partial U(x,t)}{\partial t} = \dfrac{\lambda}{x}\dfrac{\partial}{\partial x}\left[x\dfrac{\partial U(x,t)}{\partial x}\right] \\ x = R_2 & : \quad U(x,t) = 1 \\ x = R_1 & : \quad -\lambda\dfrac{\partial U(x,t)}{\partial x} = 0 \\ t = 0 & : \quad U(x,0) = 0 \end{cases} \tag{8}$$

In Figure 1 the sensitivity profiles (solution of problem (8)) for times 50, 100, 150 and 250 s are shown. Figure 2 illustrates the temperature profiles for the same times and T_b = 100°C, while in Figure 3 - the temperature profiles for boundary temperature T_b = 150°C. The lines correspond to the direct solution obtained for the new boundary temperature, the symbols correspond to the solution resulting from the Taylor formula under the assumption that T_b = 100°C, ΔT_b = 50 K.

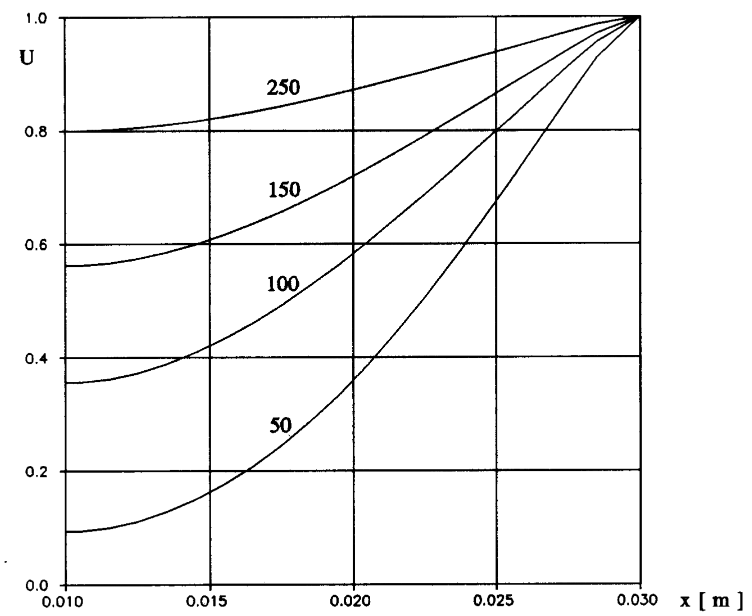

Fig. 1. Sensitivity profiles

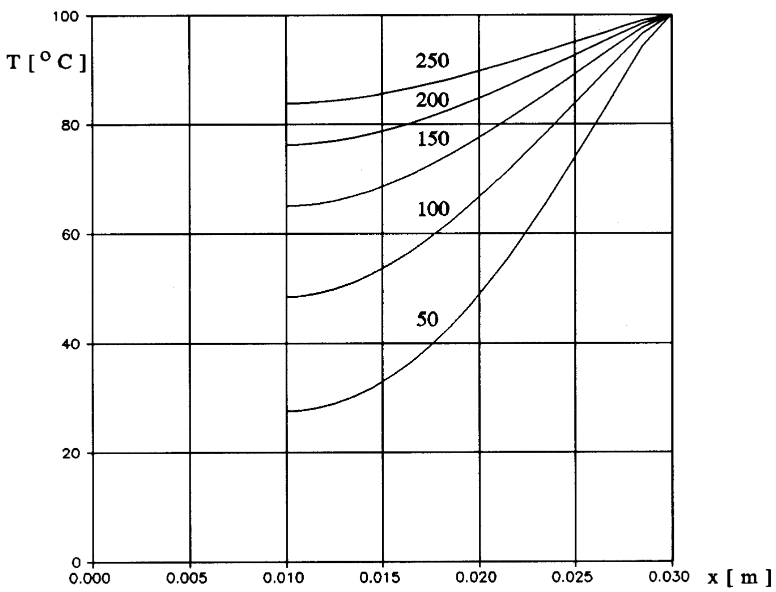

Fig. 2. Temperature profiles for $T_b = 100°C$

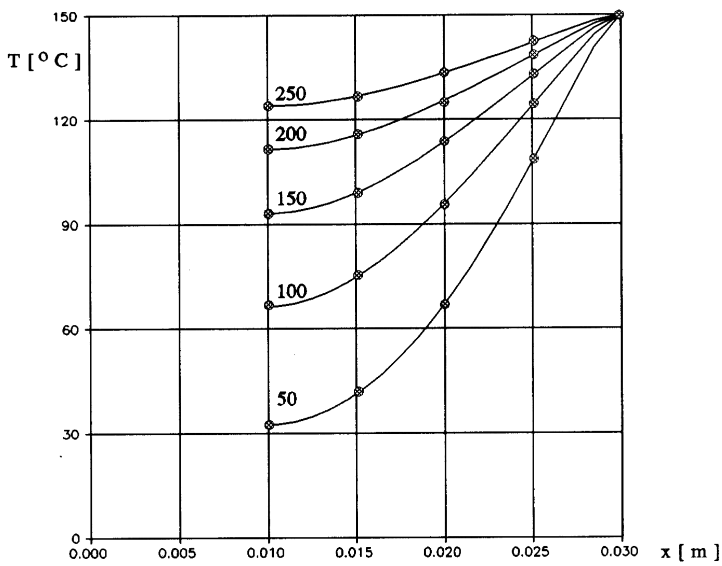

Fig. 3. Temperature profiles for $T_b = 150°C$

According to the theoretical consideration the direct and resulting from the sensitivity approach solutions are the same. In [6] the similar example concerning 2D problem can be found.

3. Sensitivity model coupled with the basic one

The heat conduction problem described by equation (1) and conditions (2) and (3) is considered again. The sensitivity of temperature with respect to volumetric specific heat and thermal conductivity is analyzed. Differentiation of the Fourier equations with respect to c gives

$$x \in \Omega \quad : \quad \frac{\partial T(x,t)}{\partial t} + c\frac{\partial U(x,t)}{\partial t} = \lambda \nabla^2 U(x,t) \tag{9}$$

Introducing the artificial source term $Q = -\partial T/\partial t$ we have

$$x \in \Omega \quad : \quad c\frac{\partial U(x,t)}{\partial t} = \lambda \nabla^2 U(x,t) + Q(x,t) \tag{10}$$

The last equation corresponds to the diffusion one with non-zero source term (see (1)). Differentiating with respect to λ we obtain

$$x \in \Omega \quad : \quad c\frac{\partial U(x,t)}{\partial t} = \nabla^2 T(x,t) + \lambda \nabla^2 U(x,t) \tag{11}$$

ore using once again the equation (1)

$$x \in \Omega \quad : \quad c\frac{\partial U(x,t)}{\partial t} = \lambda \nabla^2 U(x,t) + \frac{c}{\lambda}\frac{\partial T(x,t)}{\partial t} \tag{12}$$

which is very similar to (9).

The boundary and initial conditions assumed previously reduce (in the case of sensitivity with respect to c) to the zero conditions. In spite of this the solution concerning the function U is no-zero because of source term in the diffusion equation. In the case of sensitivity with respect to λ the Neumann condition concerning the boundary value of normal derivative leads to the formula

$$x \in \Delta\Gamma_0 \quad : \quad -\lambda\frac{\partial U(x,t)}{\partial n} - \frac{\partial T(x,t)}{\partial n} = 0 \tag{13}$$

or

$$x \in \Delta\Gamma_0 \quad : \quad -\lambda\frac{\partial U(x,t)}{\partial n} = \frac{\partial T(x,t)}{\partial n} = -\frac{q_b}{\lambda} = q_{U_b} \tag{14}$$

where q_b is the given boundary heat flux. Finally we obtain the Neumann condition with new flux denoted as q_{Ub}.

As the example of sensitivity coupled with the basic solution the following 1D task will be presented. We consider the plate (L = 5 cm), while T_0 = 100°C, λ = 1.3 W/mK, c = 1.6 MJ/m^3 K. The heat exchange between domain and environment for $x = L$ is determined by the Robin condition (ambient temperature

T_a = 0°C, heat transfer coefficient α = 100 W/m²K). For x = 0 the no-flux condition is assumed. The analysis concerns the sensitivity with respect to c. It should be pointed out that the differentiation of the Robin condition with respect to c gives the same (from the mathematical point of view) form, namely

$$x \in \Delta\Gamma_0 \quad : \quad -\lambda \frac{\partial U(x,t)}{\partial n} = \alpha\left[U(x,t) - U_a\right] \tag{15}$$

where $U_a = 0$.

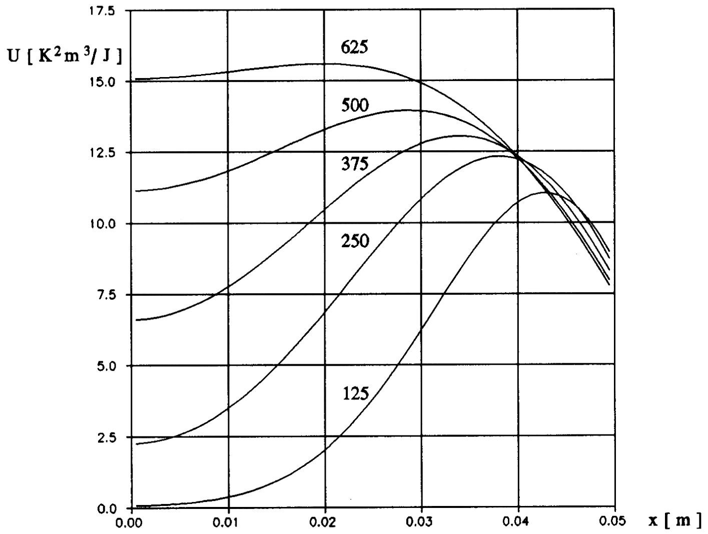

Fig. 4. Sensitivity with respect to c

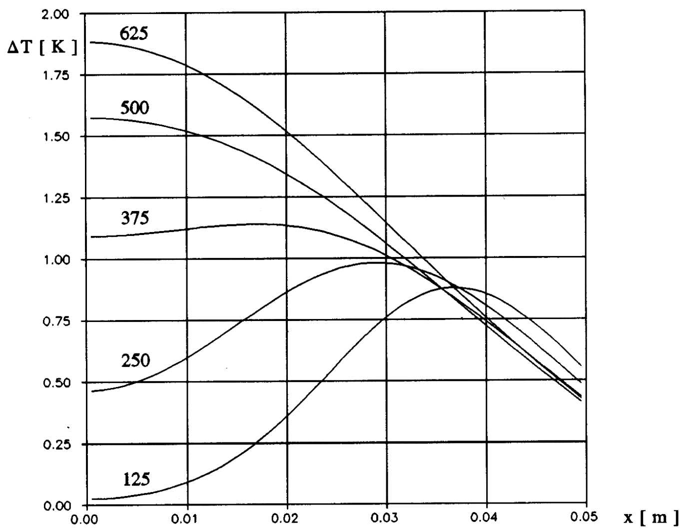

Fig. 5. Error profiles for perturbation +30%

In Figure 4 the sensitivity profiles for t = 125, 250, 375, 500 and 625 s are shown. The obtained results have been applied to rebuild the basic solution obtained for c on the solution for 1.1c, 1.2c and 1.3c. The Taylor formula of type (6) has been used, but the results are not exact because the second order sensitivity is non-zero function.

In Figure 5 the error of 'new' temperature identification corresponding to 30% perturbation of c is shown

The problem considered is linear and then the errors resulting from the simplified form of Taylor formula are rather small. In the case of non-linear problems [7-11], the errors are bigger.

4. Non-zero second order sensitivity

As the example of problem discussed the sensitivity analysis of the Robin condition parameters will be shown. The solution for disturbed values of heat transfer coefficient α and ambient temperature T_a can be found using the formula

$$\begin{gathered} T\left(x, t, \alpha \pm \Delta\alpha, T_a \pm \Delta T_a\right) = T\left(x, t, \alpha, T_a\right) \pm \\ U_\alpha \, \Delta\alpha \pm U_a \, \Delta T_a + U_{\alpha\alpha} \frac{\Delta\alpha^2}{2} + U_{aa} \frac{\Delta T_a^2}{2} \pm U_{\alpha\alpha} \, \Delta\alpha \, \Delta T_a \end{gathered} \tag{16}$$

So, the basic solution and 5 additional boundary-initial problems concerning the first and the second order sensitivities must be solved. For example the sensitivity U_α results from the following system of equations

$$\begin{cases} x \in \Omega & : \quad c \dfrac{\partial U_\alpha(x,t)}{\partial t} = \lambda \nabla^2 U_\alpha(x,t) \\ x \in \Gamma_0 & : \quad -\lambda \dfrac{\partial U_\alpha(x,t)}{\partial n} = \alpha\left[U_\alpha(x,t) - U_\alpha^a\right] \quad , \quad U_\alpha^a = \dfrac{T_a - T(x,t)}{\alpha} \\ t = 0 & : \quad U_\alpha(x,0) = 0 \end{cases} \tag{17}$$

while $U_{\alpha\alpha} = V_\alpha$ corresponds to the solution of the problem

$$\begin{cases} x \in \Omega & : \quad c \dfrac{\partial U_{\alpha\alpha}(x,t)}{\partial t} = \lambda \nabla^2 U_{\alpha\alpha}(x,t) \\ x \in \Gamma_0 & : \quad -\lambda \dfrac{\partial U_{\alpha\alpha}(x,t)}{\partial n} = \alpha\left[U_{\alpha\alpha}(x,t) - U_{\alpha\alpha}^a\right] \quad , \quad U_{\alpha\alpha}^a = \dfrac{2U_\alpha}{\alpha} \\ t = 0 & : \quad U_{\alpha\alpha}(x,0) = 0 \end{cases} \tag{18}$$

Let us consider the following 1D task (plate of thickness 6 cm and parameters λ = = 1 W/mK, c = 1.6 MJ/m^{3}K). On the outer surface α = 100 W/m^{2}K, T_a = 30^{o}C [6].

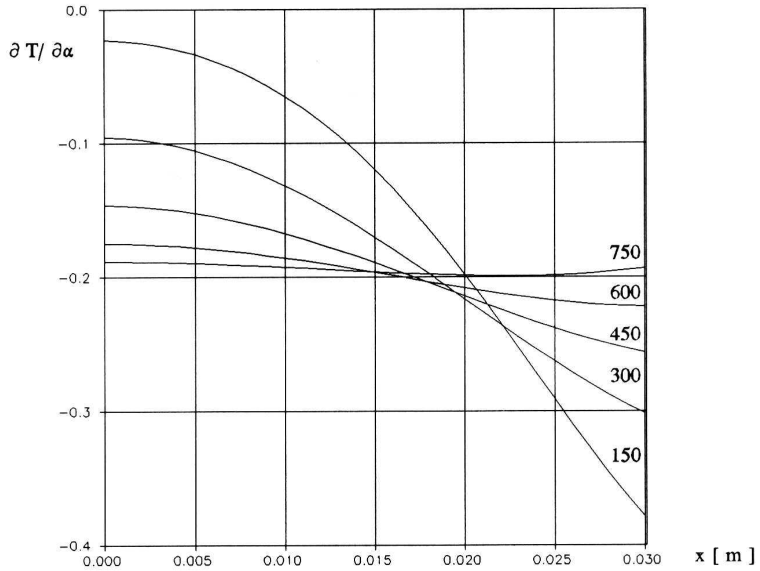

Fig. 6. First order sensitivity with respect to α

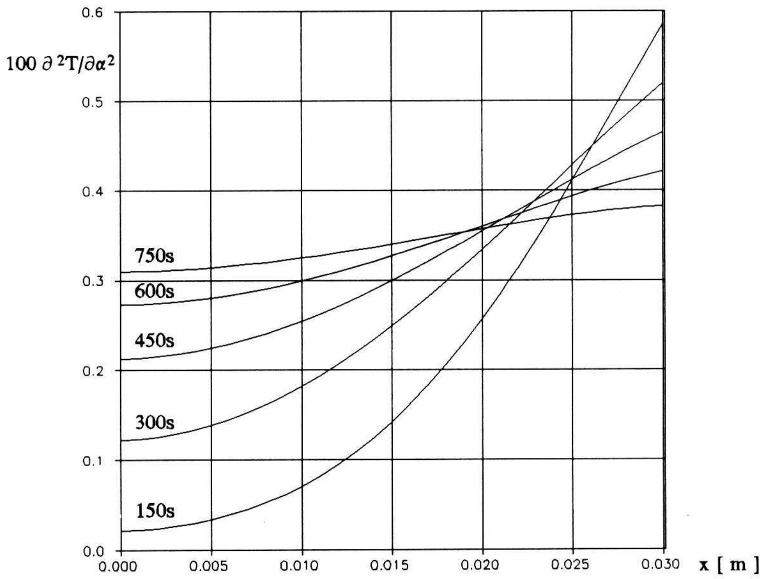

Fig. 7. Second order sensitivity with respect to α

In Figures 6 and 7 the courses of sensitivities U_α and $U_{\alpha\alpha}$ for times 150, 300, 450, 600 i 750 s are shown. One can see that sensitivity U_α is negative (the bigger value of α causes the drop of temporary temperatures in Ω). The second order

sensitivity $U_{\alpha\alpha}$ is small, but can be taken into account. The obtained sensitivity distributions U_{α}, $U_{\alpha\alpha}$, U_{a}, U_{aa} and $U_{a\alpha}$ have been used in order to find the solution for $\alpha = 70$ W/m^{2}K, $T_a = 10^{\circ}$C. The results, as it was expected, were practically the same as the solution of direct problem with new parameters α and T_a.
Summing up, in the general case the application of the first order sensitivity analysis can be not sufficient and then the second order sensitivity can be taken into account.

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