USING THE EULER'S METHOD TO SOLVE ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH A MIXTURE OF INTEGER AND CAPUTO DERIVATIVES

Tomasz Blaszczyk, Jacek S. Leszczynski

Institute of Mathematics and Computer Science, Czestochowa University of Technology, Poland
email: tomblaszczyk@gmail.com, jaclesz@gmail.com

Abstract. In this paper we present an application of the Euler's method to the numerical solution of fractional ordinary differential equations. These equations include both a classical differential operator of integer order and the fractional one defined in the Caputo sense. Our previous work was limited to the order of fractional derivative $\alpha \in \langle 0,1)$. This study considers numerical schemes for higher orders of a fractional derivative. We then compare our schemes with analytical solutions in order to show their good numerical precision.

Introduction

A contemporary mathematical modeling uses fractional ordinary differential equations as an alternative approach to the ordinary differential equations of integer order. Here we start with a class of ordinary differential equations defined as:

$$f\left(x, y(x), D^1 y(x), ..., D^p y(x), D^{\alpha_1} y(x), ..., D^{\alpha_m} y(x)\right) = 0 \tag{1}$$

where $y(x)$ is a continuous function being a solution of above equation, $D^1 y(x), ..., D^p y(x)$ are derivatives of integer order and $D^{\alpha_1} y(x), ..., D^{\alpha_m} y(x)$ are fractional derivatives of real orders $\alpha_1, ..., \alpha_m \in R$. It should be noted that an analytical solution of such equation is limited to its linear form and includes some special functions. However, numerical techniques presented in literature [1-5] have also many disadvantages, i.e. introduction of the initial conditions included in the Riemann-Liouville derivative [8], the unreasonable assumption that a method applied to a single term equation is proper for solving a multi-term equation [3, 7] etc.

In our work [8] we introduced a novel numerical technique in order to omit these disadvantages. However the technique has considered six types of equations in which the integer order belonged to the range $p \in \{0,1,2\}$ and the fractional one was $\alpha \in \langle 0,1 \rangle$. We also checked three numerical techniques applied to solution of Eqn. (1) - the Adams method [9], the Gear's method [10] and the Euler's method [9] - and four discrete forms of the left-side Caputo derivative. According to [11] we define such operator as:

$$ {}_{x_0}^{C}D_x^{\alpha} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_{x_0}^{x} \frac{y^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau \tag{2} $$

where $n = [\alpha] + 1$ and [·] indicates an integer part of a real number. In literature [2, 6, 12] one may find another discrete forms of the Caputo derivative (2).

To extend our considerations we will focus on ordinary differential equations with a mixture of integer and fractional derivatives where the real order α of a fractional derivative is greater than $\alpha \in \langle 0,1 \rangle$. We then will show and test numerical algorithms adopted to fractional differential equations where real order of the fractional derivative satisfies the range $\alpha \in \langle 1,2 \rangle$.

1. Problem statement

Here we focus on the following types of equations:

- $p > n$ for $p = 3$, $\alpha \in \langle 1,2 \rangle$, $n = 2$

$$ D^3 y(x) + \lambda \, {}_{x_0}^{C}D_x^{\alpha} y(x) = 0 \tag{3} $$

- $p = n$ for $p = 2$, $\alpha \in \langle 1,2 \rangle$, $n = 2$

$$ D^2 y(x) + \lambda \, {}_{x_0}^{C}D_x^{\alpha} y(x) = 0 \tag{4} $$

It should be noted that the function $y(x)$ being solution of such equations belongs to the class of continuous functions. To solve above equations numerically we use the Euler's method where a discrete form of the Caputo derivative (2) is proposed. In the next section we show a way how to discretize this derivative.

1.1. Analytical solution

In order to compare numerical results we solve the above system of equations analytically. In literature [12-14] one may find a way how to generate the analytical solution for some types of fractional differential equations. Here we can apply the Laplace transform [15] that to solve the equations (3) and (4) respectivelly.

Notice that the classical transform of any differential operator of integer order $m \in N$ has the following form:

$$\ell\left[D^m y(x)\right] = s^m F(s) - \sum_{k=0}^{m-1} s^k D^{m-k+1} y(x_0) \tag{5}$$

However the Laplace transform of the Caputo derivative (2) is:

$$\ell\left[{}_{x_0}^{C}D_x^{\alpha} y(x)\right] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} D^k y(x_0) \tag{6}$$

Assuming initial conditions as:

$$y(x_0) = y_0,\ D^1 y(x_0) = y'_0,\ D^2 y(x_0) = y''_0 \tag{7}$$

and using the Laplace transform and retransforming results we obtain analytical solution of Eqn. (3) in the following form:

$$\begin{aligned}
y(x) &= y_0 + y'_0(x-x_0) + y''_0(x-x_0)^2 E_{3-\alpha,3}\left(-\lambda(x-x_0)^{3-\alpha}\right) \\
D^1(x) &= y'_0 + 2y''_0(x-x_0) E_{3-\alpha,3}\left(-\lambda(x-x_0)^{3-\alpha}\right) + \\
&\quad + y''_0(3-\alpha)(x-x_0)^2 E^{(1)}_{3-\alpha,3}\left(-\lambda(x-x_0)^{3-\alpha}\right) \\
D^2(x) &= 2y''_0 E_{3-\alpha,3}\left(-\lambda(x-x_0)^{3-\alpha}\right) + 4y''_0(3-\alpha)(x-x_0) E^{(1)}_{3-\alpha,3}\left(-\lambda(x-x_0)^{3-\alpha}\right) + \\
&\quad + y''_0(3-\alpha)(x-x_0)^2 E^{(2)}_{3-\alpha,3}\left(-\lambda(x-x_0)^{3-\alpha}\right)
\end{aligned} \tag{8}$$

where $E_{\alpha,\beta}\left(-\lambda x^{\alpha}\right)$ denotes the Mittag-Leffler function [14] defined as:

$$E_{\alpha,\beta}\left(-\lambda x^{\alpha}\right) = \sum_{i=0}^{\infty} \frac{(-\lambda)^i x^{\alpha i}}{\Gamma(\alpha i + \beta)} \tag{9}$$

and $E^{(1)}_{\alpha,\beta}\left(-\lambda x^{\alpha}\right)$, $E^{(2)}_{\alpha,\beta}\left(-\lambda x^{\alpha}\right)$ are first and second derivatives of the Mittag--Leffler function defined as:

$$E^{(1)}_{\alpha,\beta}\left(-\lambda x^{\alpha}\right) = \sum_{i=1}^{\infty} \frac{(-\lambda)^i i x^{\alpha i-1}}{\Gamma(\alpha i + \beta)} \tag{10}$$

$$E^{(2)}_{\alpha,\beta}\left(-\lambda x^{\alpha}\right) = \sum_{i=1}^{\infty} \frac{(-\lambda)^i i(\alpha i - 1) x^{\alpha i-2}}{\Gamma(\alpha i + \beta)} \tag{11}$$

Solving analytically Eqn. (4), where initial conditions:

$$y\left(x_0\right)=y_0,\ D^1 y\left(x_0\right)=y'_0 \tag{12}$$

are included, we have:

$$\begin{aligned} & y(x)=y_0+y'_0\left(x-x_0\right) \\ & D^1(x)=y'_0 \end{aligned} \tag{13}$$

It should be noted that above solutions arise from our assumption that the function $y(x)$ belongs to a class of continuous functions.

1.2. Numerical technique

As we remarked before there are many numerical approaches which one uses with their advantages and disadvantages for solution of fractional differential equations. We also found in literature [2, 6, 12] many discrete forms of the Caputo derivative (2). On the base of our previous experience [8] we decided to apply the Euler's method as a method solving ordinary differential equations. Moreover we choosen the left-side form as a discrete form of the Caputo derivative (2).

Here we present the discrete form and two algorithms. Let us consider an independent value x which occurs on a length of calculations $\left\langle x_0, x_N\right\rangle$, where x_0 and x_N are the beginning and the end of the range respectivelly. We divided the range on the N-parts and we obtain a homogeneous grid $x_0<x_1<\ldots<x_N$. Then the Caputo derivative (2) has the following discrete form:

$${}_{x_0}^{C}D_x^\alpha y(x) \cong \frac{1}{\Gamma(n-\alpha+1)} \sum_{k=1}^N B_{k-1}\left[\left(x_N-x_{k-1}\right)^{n-\alpha}-\left(x_N-x_k\right)^{n-\alpha}\right] \tag{14}$$

where $x \in\left\langle x_{k-1}, x_k\right\rangle$ and $D^n y\left(x_{k-1}\right)=B_{k-1}$.

The Euler's method [9] is a forward one-step method. Using this method one may obtain the numerical scheme for an ordinary differential equation of the first order as:

$$y_k=y_{k-1}+h f\left(x_{k-1}, y_{k-1}\right),\ k=1, \ldots, N \tag{15}$$

In this point of our considerations we propose two algorithms which solve the above differential equations.

Algorithm 1

Eqn. (3) with initial conditions (7) is solved by the following algorithm

step 1 Preparation of necessary data: initial conditions, the fractional order $\alpha \in \langle 1,2)$, the total length of calculations $x \in \langle x_0, x_N \rangle$ and the step of calculations *h*.
step 2 Governing calculations: let $k = 1,...,N$ then:

$$\begin{aligned} &{}_{x_0}^{C}D_x^{\alpha} y_k = \frac{1}{\Gamma(3-\alpha)} \sum_{j=1}^{k} D^2 y_{k-1} \left[\left(x_k - x_{j-1}\right)^{2-\alpha} - \left(x_k - x_j\right)^{2-\alpha} \right] \\ &y_k = y_{k-1} + hD^1 y_{k-1} \\ &D^1 y_k = D^1 y_{k-1} + hD^2 y_{k-1} \\ &D^2 y_k = D^2 y_{k-1} - h\lambda\, {}_{x_0}^{C}D_x^{\alpha} y_k \end{aligned} \tag{16}$$

Algorithm 2
Considering Eqn. (4) with initial conditions (12) we obtain the following algorithm
step 1 Preparation of necessary data: initial conditions, the fractional order $\alpha \in \langle 1,2)$, the total length of calculations $x \in \langle x_0, x_N \rangle$, the step of calculations *h* and additionally $D^2 y_0 = 0$.
step 2 Governing calculations: let $k = 1,...,N$ then:

$$\begin{aligned} &{}_{x_0}^{C}D_x^{\alpha} y_k = \frac{1}{\Gamma(3-\alpha)} \sum_{j=1}^{k} D^2 y_{k-1} \left[\left(x_k - x_{j-1}\right)^{2-\alpha} - \left(x_k - x_j\right)^{2-\alpha} \right] \\ &y_k = y_{k-1} + hD^1 y_{k-1} \\ &D^1 y_k = D^1 y_{k-1} - h\lambda\, {}_{x_0}^{C}D_x^{\alpha} y_k \\ &D^2 y_k = -\lambda\, {}_{x_0}^{C}D_x^{\alpha} y_k \end{aligned} \tag{17}$$

In summary we proposed some numerical schemes suitable for two types of fractional differential equations (3) and (4).

2. Results and discussion

In this section we illustrate how algorithms operate in practice. First we compare numerical results with the analytical ones. Let us to solve Eqn. (3) where $\lambda = 1$ and initial conditions are determined as:

$$y(0) = D^1 y(0) = 1,\; D^2 y(0) = 1 \tag{18}$$

We use algorithm 1 in order to solve numerically Eqn. (3) with initial conditions (18). Notice that this equation has analytical solution presented by formula (8). Tables 1 and 2 show analytical values (8) at assumed points and the difference between the numerical and analytical results being measure of an absolute error. The tables differ by the assumed step of calculations.

Table 1

Analytical results of Eqn. (3) with initial condition (18) and errors generated by the Euler's method for h = 0.01

	y(1)	y(4)	y(6)	y(8)	y(10)
$\alpha = 1.1$					
Analytical	2.4535821	6.8069550	7.6912657	10.2724904	12.7546663
Error	4.11e-3	2.46e-3	1.10e-3	2.75e-3	9.38e-4
$\alpha = 1.5$					
Analytical	2.4218511	7.4020740	9.8036872	12.1712704	14.5673990
Error	3.60e-3	1.56e-3	8.79e-4	1.00e-3	9.21e-4
$\alpha = 1.9$					
Analytical	2.3795378	7.9058846	11.5808691	15.1697472	18.7034378
Error	2.48e-3	2.20e-3	2.17e-3	2.19e-3	2.20e-3

Table 2

Analytical results of Eqn. (3) with initial condition (18) and errors generated by the Euler's method for h = 0.005

	y(1)	y(4)	y(6)	y(8)	y(10)
$\alpha = 1.1$					
Analytical	2.4535821	6.8069550	7.6912657	10.2724904	12.7546663
Error	2.05e-3	1.23e-3	5.52e-4	1.39e-3	4.73e-4
$\alpha = 1.5$					
Analytical	2.4218511	7.4020740	9.8036872	12.1712704	14.5673990
Error	1.81e-3	7.79e-4	4.30e-4	5.03e-4	4.62e-4
$\alpha = 1.9$					
Analytical	2.3795378	7.9058846	11.5808691	15.1697472	18.7034378
Error	1.27e-3	1.17e-3	1.15e-3	1.15e-3	1.15e-3

Analyzing above tables we can observe that errors generated by the algorithm are independent of the point *x* taken from the range $\langle 0,10 \rangle$ where the function value $y(x)$ is calculated. It should be noted that twofold decrease of the calculation step *h* influences linearly to a decrease of the error.

Let us take into consideration Eqn. (4), where $\lambda = 1$ and initial conditions are:

$$y(0) = D^1 y(0) = 1 \tag{19}$$

Notice that the analytical solution (13) for such equation is independent on the parameter α. Using algorithm 2 we can observe that the numerical solution does not generate any errors. Therefore we omit tables and show results graphically.

Figures 1 and 2 present some comparison between analytical and numerical results.

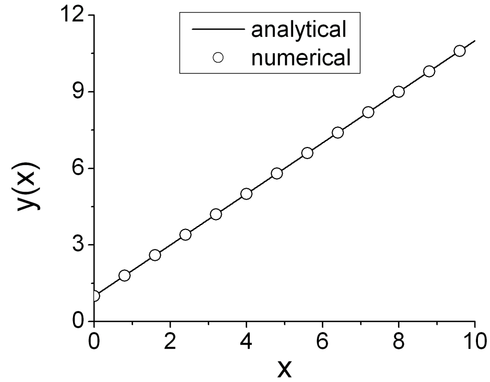

Fig. 1. Comparison of analytical and numerical results of the function $y(x)$ being the solution of Eqn. (4) with initial conditions (19)

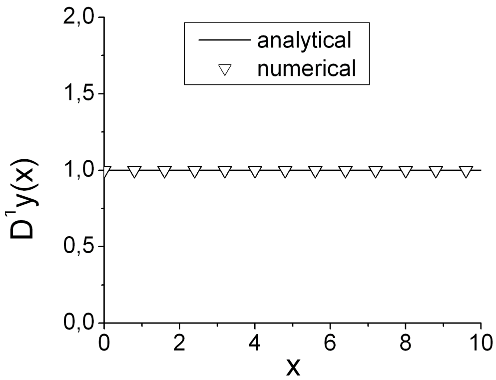

Fig. 2. Comparison of analytical and numerical results of the first derivative $D^1y(x)$ being the solution of Eqn. (4) with initial conditions (19)

Analyzing such figures we can see that numerical results exactly fit the analytical one.

The last case concerns the analytical solution (8) of Eqn. (3) with initial conditions (18). Figures 3, 4 and 5 show solutions for different values of the parameter α.

Analyzing results presented by Figure 3 we can observe interesting behaviour of the function $y(x)$ over the independent value x for different values of the parameter α. When the parameter α tends to two then the function behaves almost linearly.

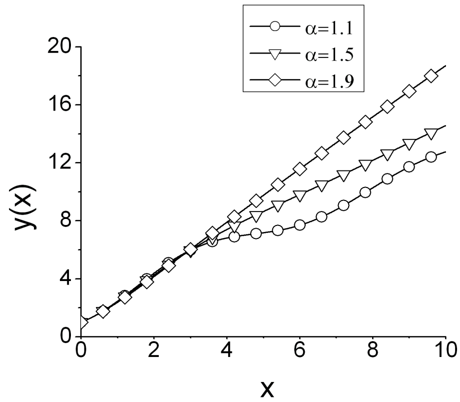

Fig. 3. Analytical results $y(x)$ of Eqn. (3) with initial conditions (18) being dependent on the parameter α

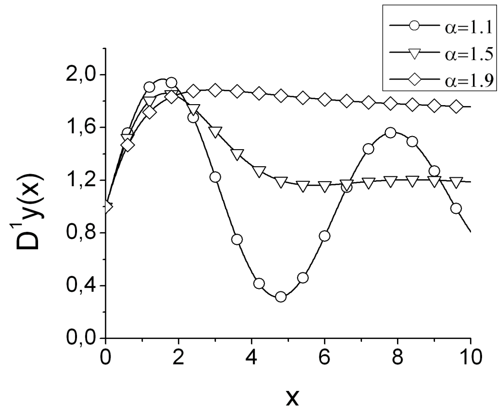

Fig. 4. Analytical results $D^1 y(x)$ of Eqn. (3) with initial conditions (18) for different values of the parameter α

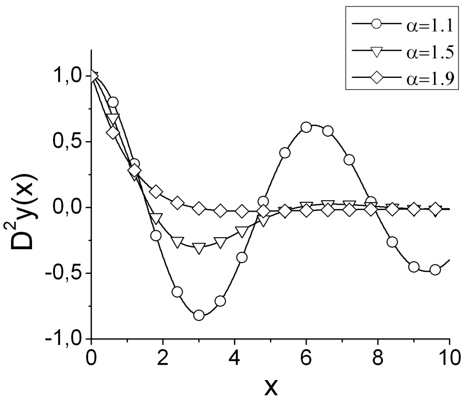

Fig. 5. Analytical results $D^2 y(x)$ of Eqn. (3) with initial conditions (18) for different values of the parameter α

Summarising our considerations we proved that the Euler's method is good for the numerical solution of ordinary differential equations where an arbitrary positive value of the Caputo derivative is assumed.

Conclusions

In this paper we presented an extension of numerical algorithms to solve ordinary differential equations for arbitrary positive values of the real order α of the Caputo fractional derivative. On the base of previous results [8] we modified the discrete form of the Caputo derivative being dependent on a range of the parameter α. When the range increases then a number of discrete equations occuring in the algorithm is increases too. Taking into consideration both the complexity of numerical schemes and the errors generated by the numerical method we recommend the Euler's method as a good method for numerical treatment of fractional ordinary differential equations.

References

[1] Blank L., Numerical treatment of differential equations of fractional order, Numerical Analysis Report 287, Manchester Centre for Numerical Computational Mathematics 1996.

[2] Diethelm K., Ford N.J., Freed A.D., Luchko Y., Algorithms for the fractional calculus: A selection of numerical methods, Computer Methods in Applied Mechanics and Engineering 2005, 194, 743-773.

[3] Diethelm K., Luchko Y., Numerical solution of linear multi-term differential equations of fractional order, Technical report, TU Braunschweig 2001.

[4] Ford N.J., Connolly J.A., Comparison of numerical methods for fractional differential equations, CPAA 2006, 5, 289-307.

[5] Heymans N., Podlubny I., Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, Rheologica Acta 2006, 45, 765-771(7).

[6] Gorenflo R., Mainardi F., Moretti D., Pagnini G., Paradisi P., Discrete random walk models for space-time fractional diffusion, Chemical Physics 2002, 284, 521-541.

[7] El-Sayed A.M.A., El-Mesiry A.E.M., El-Saka H.A.A., Numerical solution for multi-term fractional (arbitrary) orders differential equations, Computational and Applied Mathematics 2004, 23, 33-54.

[8] Leszczynski J.S., Blaszczyk T., A novel numerical techique used in the solution of ordinary differential equations with a mixture of integer and fractional derivatives, Numerical Algorithms 2007 (in print).

[9] Palczewski A., Ordinary differential equations: theory and numerical methods, WNT, Warsaw 1999 (in Polish).

[10] Gear C.W., Numerical initial value problems in ordinary differential equation, Prentice-Hall, Englewood Cliffs 1971.

[11] Caputo M., Linear models of dissipation whose Q is almost frequency independent, Part II, Geophys. J. R. Astr. Soc. 1967, 13, 529-539.

[12] Oldham K.B., Spanier J., The fractional calculus. Theory and applications of differentiation and integration to arbitrary order, Academic Press, New York 1974.

[13] Podlubny I., Fractional Differential Equations, Academic Press, San Diego 1999.

[14] Samko S.G., Kilbas A.A., Marichev O.I., Integrals and derivatives of fractional order and same of their applications, Gordon and Breach, London 1993.

[15] Miller K.S., Ross B., An introduction to the fractional differential equations, Wiley and Sons, New York 1993.