NUMERICAL SOLUTION OF BIOHEAT TRANSFER EQUATION BY MEANS OF THE DUAL RECIPROCITY BEM

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Abstract. Application of the standard boundary element method for numerical solution of the bioheat transfer equation requires discretization not only the boundary but also the interior of the domain considered. It results from the presence of internal heat sources in the biological tissue (metabolic and perfusion sources). In this paper the variant of the BEM which is connected only with the boundary discretization is presented. It is the essential advantage of the algorithm proposed in comparison with the classical one. As example, the problem of temperature field computations in heating biological tissue domain is solved.

Introduction

The blood perfusion greatly affects the thermal response of living tissue and Pennes proposed quantifying heat transfer effects in perfused biological tissue by a heat source appearing in the energy equation [1-3]. The capacity of internal heat sources is proportional to the perfusion rate and the difference between the tissue temperature and the global arterial blood temperature.

In this paper we consider the steady state problem and then the Pennes equation takes a form

$$x \in \Omega : \ \lambda \ \nabla^2 T(x) + c_B \ G_B \left[T_B - T(x) \right] + Q_{met} = 0 \qquad (1)$$

where λ [W/ mK] is the thermal conductivity, Q_{met} [W/m^3] is the metabolic heat source, G_B [m^3/s/m^3 tissue] is the blood perfusion rate, c_B [J/m^3 K] is the volumetric specific heat of blood, T_B is the arterial blood temperature, T denotes the temperature, $x = \{x_1, x_2\}$.
The equation (1) can be written as follows

$$x \in \Omega : \ \lambda \ \nabla^2 T(x) - g T(x) + Q = 0 \qquad (2)$$

where

$$g = G_B c_B, \quad Q = g T_B + Q_{met} \qquad (3)$$

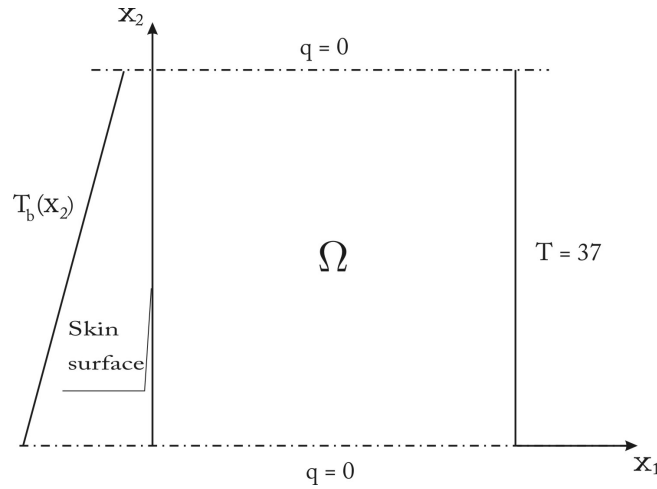Equation (2) is supplemented by the boundary conditions shown in Figure 1.

Fig. 1. Domain considered and boundary conditions

On the skin surface the Dirichlet condition is assumed

$$0 < x_2 < L: \quad T(0, x_2) = T_b(x_2) = T_{max} - \frac{T_{max} - 37}{L} x_2 \tag{4}$$

where T_{max} is the maximum value of temperature. On the upper and lower surfaces the zero heat flux $q(x) = -\lambda \, \partial T(x)/\partial n$ can be accepted.

The problem formulated is solved by means of the dual reciprocity boundary element method.

1. Boundary element method for the bioheat transfer equation

The standard boundary element method algorithm leads to the following integral equation [4-6]

$$B(\xi)T(\xi) + \int_\Gamma T^*(\xi, x) \, q(x) \, \mathrm{d}\Gamma =$$

$$= \int_\Gamma q^*(\xi, x) \, T(x) \, \mathrm{d}\Gamma + \int_\Omega \left[Q - g \, T(x) \right] T^*(\xi, x) \, \mathrm{d}\Omega \tag{5}$$

where ξ is the observation point, $B(\xi) \in (0, 1]$, $T^*(\xi, x)$ is the fundamental solution, $q(x) = -\lambda \, \partial T(x)/\partial n$ is the heat flux, $q^*(\xi, x) = -\lambda \, \partial T^*(\xi, x)/\partial n$ is the heat flux resulting from the fundamental solution.

Fundamental solution has the following form

$$T^*\left(\xi, x\right) = \frac{1}{2\pi\lambda}\ln\frac{1}{r} \tag{6}$$

where r is the distance between the points ξ and x. Heat flux resulting from the fundamental solution can be calculated analytically and then

$$q^*\left(\xi, x\right) = \frac{d}{2\pi r^2} \tag{7}$$

where

$$d = \left(x_1 - \xi_1\right)\cos\alpha_1 + \left(x_2 - \xi_2\right)\cos\alpha_2 \tag{8}$$

while $\cos\alpha_1$, $\cos\alpha_2$ are the directional cosines of the boundary normal vector n.

It should be pointed out that the function $T^*\left(\xi, x\right)$ fulfills the equation

$$\lambda\,\nabla^2 T\left(x\right) = -\delta\left(\xi, x\right) \tag{9}$$

where $\delta\left(\xi, x\right)$ is the Dirac function.

2. DRBEM for the bioheat transfer equation

The solution of Pennes equation (2) can be written as a sum

$$T\left(x\right) = \hat{T}\left(x\right) + U\left(x\right) \tag{10}$$

where the first function is the solution of Laplace'a equation

$$x \in \Omega: \quad \lambda\nabla^2\hat{T}\left(x\right) = 0 \tag{11}$$

and $U\left(x\right)$ is a particular solution of equation

$$x \in \Omega: \quad \lambda\nabla^2 U\left(x\right) - g\,U\left(x\right) = 0 \tag{12}$$

From equations (2), (11), (12) results that

$$T\left(x\right) = U\left(x\right) + \frac{Q}{g} \tag{13}$$

It is generally difficult to find the solution $U(x)$. In the dual reciprocity method the following approximation is proposed [5]

$$U(x) \approx \sum_{k=1}^{N+L} a_k f_k(x) \tag{14}$$

where a_k are unknown coefficients and $f_k(x)$ are approximating functions fulfilling the equations

$$\frac{\lambda}{g} \nabla^2 U_k(x) = f_k(x) \tag{15}$$

Putting (15) into (14) one obtains

$$U(x) = \frac{\lambda}{g} \sum_{k=1}^{N+L} a_k \nabla^2 U_k(x) \tag{16}$$

In equations (14), (16) $N+L$ corresponds to the total number of the nodes, where N is the number of boundary nodes and L is the number of internal nodes. We consider the last integral in equation (5)

$$D = \int_{\Omega} \left[Q - g T(x) \right] T^*(\xi,x) \, d\Omega = -g \int_{\Omega} U(x) T^*(\xi,x) \, d\Omega \tag{17}$$

Taking into account the dependence (16) one obtains

$$D = -\sum_{k=1}^{N+L} \int_{\Omega} \left[a_k \, \lambda \, \nabla^2 U_k(x) \right] T^*(\xi,x) \, d\Omega \tag{18}$$

Using the second Green formula [2, 4] one has

$$D = -\sum_{k=1}^{N+L} a_k \int_{\Omega} \left[\lambda \, \nabla^2 T^*(\xi,x) \right] U_k(x) \, d\Omega -$$

$$\sum_{k=1}^{N+L} a_k \int_{\Gamma} \left[\lambda \, T^*(\xi,x) \frac{\partial U_k(x)}{\partial n} - \lambda \, U_k(x) \frac{\partial T^*(\xi,x)}{\partial n} \right] d\Gamma \tag{19}$$

Because (c.f. formula (9))

$$\int_{\Omega} \left[\lambda \, \nabla^2 T^*(\xi,x) \right] U_k(x) \, d\Omega =$$

$$-\int_{\Omega} \delta(\xi,x) U_k(x) \, d\Omega = -B(\xi) U_k(\xi) \tag{20}$$

so

$$D = \sum_{k=1}^{N+L} a_k B(\xi) U_k(\xi) +$$

$$\sum_{k=1}^{N+L} a_k \int_{\Gamma} \left[T^*(\xi, x) W_k(x) - U_k(x) q^*(\xi, x) \right] d\Gamma \tag{21}$$

where

$$W_k(x) = -\lambda \frac{\partial U_k(x)}{\partial n} \tag{22}$$

Taking into account the formula (21) the equation (5) can be written in the form

$$B(\xi)T(\xi) + \int_{\Gamma} T^*(\xi, x) q(x) d\Gamma = \int_{\Gamma} q^*(\xi, x) T(x) d\Gamma +$$

$$\sum_{k=1}^{N+L} a_k \left[B(\xi) U_k(\xi) + \int_{\Gamma} T^*(\xi, x) W_k(x) d\Gamma - \int_{\Gamma} q^*(\xi, x) U_k(x) d\Gamma \right] \tag{23}$$

3. Numerical realization of the DRBEM

In order to solve the equation (23), the boundary Γ is divided into N boundary elements and in the interior of the domain L internal nodes are distinguished. In the case of constant boundary elements it is assumed that

$$x \in \Gamma_j : \quad \begin{cases} T(x) = T(x_j) = T_j \\ q(x) = q(x_j) = q_j \end{cases} \tag{24}$$

$$x \in \Gamma_j : \quad \begin{cases} U_k(x) = U_k(x_j) = U_{jk} \\ W_k(x) = W_k(x_j) = W_{jk} \end{cases} \tag{25}$$

So, the following approximation of equation (23) can be taken into account $(i = 1, 2, ..., N, N+1, ..., N+L)$

$$B_i T_i + \sum_{j=1}^{N} q_j \int_{\Gamma_j} T^*(\xi_i, x) d\Gamma_j = \sum_{j=1}^{N} T_j \int_{\Gamma_j} q^*(\xi_i, x) d\Gamma_j +$$

$$+ \sum_{k=1}^{N+L} a_k \left[B_i U_{ik} + \sum_{j=1}^{N} W_{jk} \int_{\Gamma_j} T^*(\xi_i, x) d\Gamma_j - \sum_{j=1}^{N} U_{jk} \int_{\Gamma_j} q^*(\xi_i, x) d\Gamma_j \right] \tag{26}$$

or

$$B_i T_i + \sum_{j=1}^{N} G_{ij} q_j = \sum_{j=1}^{N} \hat{H}_{ij} T_j + \sum_{k=1}^{N+L} a_k \left(B_i U_{ik} + \sum_{j=1}^{N} G_{ij} W_{jk} - \sum_{j=1}^{N} \hat{H}_{ij} U_{jk} \right) \quad (27)$$

where

$$G_{ij} = \int_{\Gamma_j} T^* (\xi_i, x) \mathrm{d}\Gamma_j \quad (28)$$

and

$$\hat{H}_{ij} = \int_{\Gamma_j} q^* (\xi_i, x) \mathrm{d}\Gamma_j \quad (29)$$

while $B_i = B(\xi_i)$. We define [5, 6]

$$U_{jk} = \frac{r_{jk}^2}{4} + \frac{r_{jk}^3}{9} \quad (30)$$

where (Figure 2)

$$r_{jk}^2 = \left(x_{1k} - x_{1j} \right)^2 + \left(x_{2k} - x_{2j} \right)^2 \quad (31)$$

Using the formula (22) one obtains

$$W_{jk} = -\lambda \begin{bmatrix} \cos\alpha_{1j} & \cos\alpha_{2j} \end{bmatrix} \begin{bmatrix} \dfrac{\partial U_{jk}}{\partial x_{1j}} \\[2mm] \dfrac{\partial U_{jk}}{\partial x_{2j}} \end{bmatrix} = \lambda d_{jk} \left(\frac{1}{2} + \frac{1}{3} r_{jk} \right) \quad (32)$$

where

$$d_{jk} = \left(x_{1k} - x_{1j} \right) \cos\alpha_{1j} + \left(x_{2k} - x_{2j} \right) \cos\alpha_{2j} \quad (33)$$

Because

$$\nabla^2 U_{sk} = \frac{\partial^2 U_{sk}}{\partial x_{1k}^2} + \frac{\partial^2 U_{sk}}{\partial x_{2k}^2} = 1 + r_{sk} \quad (34)$$

so on the basis of equation (15) one has

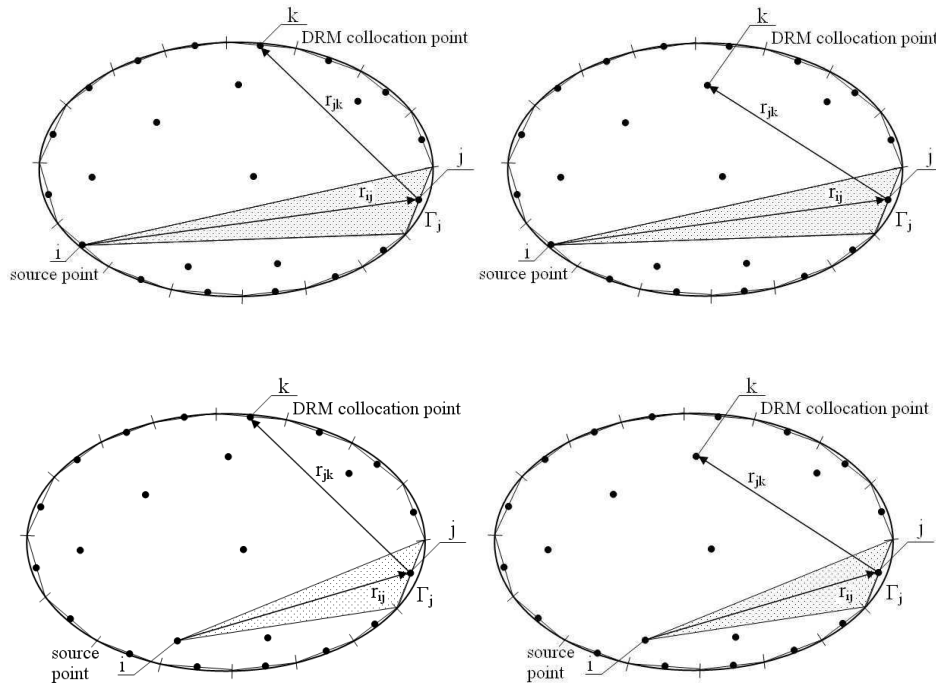$$f_{sk} = f_k (x_s) = \frac{\lambda}{g} (1 + r_{sk}) \quad (35)$$

Fig. 2. Vectors r_{ij} and r_{jk} for boundary and internal source points

Taking into account the dependencies (13), (14) one obtains

$$T_s - \frac{1}{g}Q = \sum_{k=1}^{N+L} a_k f_{sk} \tag{36}$$

where $T_s = T(x_s)$. The system of equations (36) can be written in the matrix form

$$\begin{bmatrix} T_1 - Q/g \\ T_2 - Q/g \\ \cdots \\ T_{N+L} - Q/g \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1,N+L} \\ f_{21} & f_{22} & \cdots & f_{2,N+L} \\ \cdots & \cdots & \cdots & \cdots \\ f_{N+L,1} & f_{N+L,2} & \cdots & f_{N+L,N+L} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_{N+L} \end{bmatrix} \tag{37}$$

or

$$\mathbf{T} - \mathbf{Z} = \mathbf{f}\mathbf{a} \quad \rightarrow \quad \mathbf{a} = \mathbf{f}^{-1}(\mathbf{T} - \mathbf{Z}) \tag{38}$$

where

$$\mathbf{T} = \begin{bmatrix} T_1 \\ T_2 \\ \dots \\ T_N \\ T_{N+1} \\ \dots \\ T_{N+L} \end{bmatrix}, \qquad \mathbf{Z} = \begin{bmatrix} Q/g \\ Q/g \\ \dots \\ Q/g \\ Q/g \\ \dots \\ Q/g \end{bmatrix} \tag{39}$$

The following matrices of dimensions $N+L \times N+L$ can be defined

$$\mathbf{G} = \begin{bmatrix} G_{11} & G_{12} & \dots & G_{1,N} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ G_{N,1} & G_{N,2} & \dots & G_{N,N} & 0 & \dots & 0 \\ G_{N+1,1} & G_{N+1,2} & \dots & G_{N+1,N} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ G_{N+L,1} & G_{N+L,2} & \dots & G_{N+L,N} & 0 & \dots & 0 \end{bmatrix} \tag{40}$$

$$\mathbf{H} = \begin{bmatrix} H_{11} & H_{12} & \dots & H_{1,N} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ H_{N,1} & H_{N,2} & \dots & H_{N,N} & 0 & \dots & 0 \\ H_{N+1,1} & H_{N+1,2} & \dots & H_{N+1,N} & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ H_{N+L,1} & H_{N+L,2} & \dots & H_{N+L,N} & 0 & \dots & -1 \end{bmatrix} \tag{41}$$

where

$$H_{ij} = \begin{cases} \hat{H}_{ij}, & i \neq j \\ \hat{H}_{ij} - 1/2, & i = j \end{cases} \tag{42}$$

and

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & \dots & U_{1,N} & U_{1,N+1} & \dots & U_{1,N+L} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ U_{N,1} & U_{N,2} & \dots & U_{N,N} & U_{N,N+1} & \dots & U_{N,N+L} \\ U_{N+1,1} & U_{N+1,2} & \dots & U_{N+1,N} & U_{N+1,N+1} & \dots & U_{N+1,N+L} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ U_{N+L,1} & U_{N+L,2} & \dots & U_{N+L,N} & U_{N+L,N+1} & \dots & U_{N+L,N+L} \end{bmatrix} \tag{43}$$

$$
\mathbf{W} = \begin{bmatrix} W_{11} & W_{12} & \ldots & W_{1,N} & W_{1,N+1} & \ldots & W_{1,N+L} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ W_{N,1} & W_{N,2} & \ldots & W_{N,N} & W_{N,N+1} & \ldots & W_{N,N+L} \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{44}
$$

So, the system of equations (27) can be written in the matrix form

$$
\mathbf{Gq} = \mathbf{HT} + (\mathbf{GW} - \mathbf{HU})\mathbf{a} \tag{45}
$$

or (c.f. formula (38))

$$
\mathbf{Gq} = \mathbf{HT} + (\mathbf{GW} - \mathbf{HU})\mathbf{f}^{-1}(\mathbf{T} - \mathbf{Z}) \tag{46}
$$

where

$$
\mathbf{q} = \begin{bmatrix} q_1 \\ \ldots \\ q_N \\ 0 \\ \ldots \\ 0 \end{bmatrix} \tag{47}
$$

4. Results of computations

The square of dimensions 0.03×0.03 m has been considered. The following input data have been assumed: thermal conductivity $\lambda = 0.75$ W/(m K), perfusion coefficient $g = 1998.1$ W/(m^3 K), $Q = 74349.7$ W/m^3, temperature $T_{max} = 60^{\circ}$C (c.f. Fig. 1).
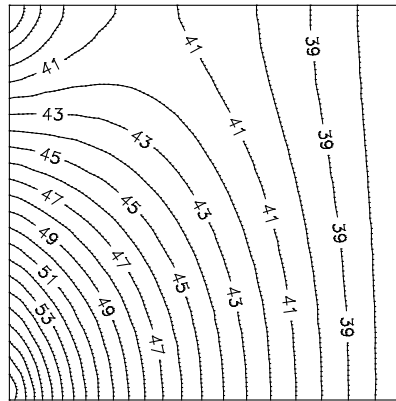

Fig. 3. Temperature distribution

The problem has been solved by means of the DRBEM. The boundary has been divided into 60 constant boundary elements, in the interior of the domain 225 internal points has been distinguished. In Figure 3 the temperature field in the domain considered is shown.

References

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