ON A CERTAIN EXTENSION OF THE PRINCIPLE OF STATIONARY ACTION AND ITS APPLICATIONS

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Abstract. The aim of this contribution is to propose a certain extension of the principle of stationary action which can be applied to derive the hyperbolic and elliptic PDEs with quasi-linear operators. The proposed generalized form of the above principle will be used to formations of mathematical models for solids and structures.

Introduction

This paper presents some from theoretical results which have been obtained recently at the Institute of Mathematics and Computer Science and Institute of Computer and Information Sciences as a part of researches on thermomechanics of microheterogeneous solids and structures. It is known that many PDEs of mathematical physics (e.g. of the parabolic type heat conduction, linear visco-elasticity, quasi-linear heat conduction) cannot be derived from the well-known principle of stationary action as the Euler-Lagrange equations, [1, 2]. Thus, the question arises how to modify the principle of stationary action in order to obtain the aforementioned equations.

In the present lecture it is proposed more general approach to the above problem which starts from a certain generalization of the principle of stationary action. To this end we introduce what will be called the extended action functional. On this basis we formulate the modified form of the principle of stationary action. The general considerations will be illustrated by some examples of applications of the above principle. As an example we can mention here the variational approach leading to the hyperbolic and elliptic PDEs with quasi-linear operators.

1. A restricted variation of integral functionals

Let W be a linear vector space and $\mathcal{A} : W^2 \ni (u,w) \to \mathcal{A}(u,w) \in R$ be a certain functional.

Now we shall introduce the notion of restricted variation of A for increment $h \in W$ by setting

$$\bar{\delta} A\ (w;\, h) \equiv \frac{d}{d\varepsilon} A\ (u, w + h\varepsilon)\Big|_{\substack{\varepsilon=0 \\ u=w}} \tag{1}$$

It means that it is variation of $A\ (u, \cdot)$ restricted by $u = w$.

In the subsequent considerations it will be assumed that $W = \left(C^1(\Omega)\right)^m$ where Ω is a regular region in R^n. Functional A will be given in the form

$$A\ (u, w) = \int_{\Omega} L\big(x;\, u(x), \nabla u(x);\, w(x), \nabla w(x)\big)\, dx, \quad x \in \bar{\Omega} \tag{2}$$

where $L, \dfrac{\partial L}{\partial w}, \dfrac{\partial L}{\partial \nabla w}$ are continuous functions.

Let us define

$$\frac{\bar{\partial} L}{\partial w} \equiv \frac{\partial L}{\partial w}\bigg|_{u=w}, \quad \frac{\bar{\partial} L}{\partial \nabla w} \equiv \frac{\partial L}{\partial \nabla w}\bigg|_{u=w} \tag{3}$$

as restricted partial derivatives of $L(\cdot)$. Hence, after some manipulations we obtain

$$\bar{\delta} A\ (w, h) = \int_{\Omega} \left[\left(\frac{\bar{\partial} L}{\partial w} - \nabla \cdot \frac{\bar{\partial} L}{\partial \nabla w} \right) h \right] dx + \nabla \cdot \int_{\Omega} \frac{\bar{\partial} L}{\partial \nabla w} \cdot h dx \quad \forall h \in W \tag{4}$$

It can be emphasized that $L(\cdot)$ can also depend on $\nabla \otimes \nabla ... \nabla w$ and $\nabla \otimes \nabla ... \nabla u$.

2. The principle of stationary extended action

Let V be a linear subspace of $W = \left(C^1(\Omega)\right)^m$ and $M = v_* + V$, $v_* \notin V$ be a linear manifold in *W*. Functional $\mathrm{A} = \mathrm{A}\,(u, w)$ where $(u, w) \in M^2$ will be called the extended action functional (the extended action *EA*). In the applications of this formalism it is assumed that the aforementioned functional determines the expected behaviour of a certain physical system.

It can be proved that necessary condition for functional A to have a minimum is

$$\bar{\delta}\mathrm{A}\left(w;\, h\right)=0,\ \ w\in M,\, \forall h\in V \tag{5}$$

The above condition will be called the principle of stationary extended action (*PSEA*). It means that *PSEA* is a principle of stationary action for the family of functionals $\mathrm{A}\left(u,\cdot\right): W \rightarrow R$ indexed by $u \in W$, under constraint $u = w$. Hence $L = L\left(x, u, \nabla u, w, \nabla w\right)$ is extended lagrangian (family of lagrangians indexed by $u \in W$).

As an example we consider a problem in which $V = \left(C_0^1\left(\Omega\right)\right)^m$, $v_* \in \left(C\left(\Omega\right)\right)^m$ where Ω is a regular region in R^n. In this case $M = v_* + V \subset\subset \left(C^1\left(\Omega\right)\right)^m \equiv W$. Applying the principle of stationary extended action (5) we obtain

$$\int_{\Omega}\left[\frac{\bar{\partial}L}{\partial w}-\nabla\cdot\left(\frac{\bar{\partial}L}{\partial\nabla w}\right)\right]h dx=0, \qquad \forall h\in\left(C_0^1\left(\Omega\right)\right)^m \tag{6}$$

which leads to equations

$$\frac{\bar{\partial}L}{\partial w}-\nabla\cdot\left(\frac{\bar{\partial}L}{\partial\nabla w}\right)=0 \tag{7}$$

Equations (7) constitute Euler-Lagrange equations of $\mathrm{A}\left(u,\cdot\right)$ constrained by $u = w$.

3. Application to problems governed by quasi-linear PDEs

Now, let us assume that extended lagrangian L is given by

$$L=\frac{1}{2}\nabla w\cdot A\left(x,u\right)\nabla w+\frac{1}{2}w\cdot B\left(x,u\right)w+C\left(x,u\right)\nabla w+d\left(x,u\right)w \tag{8}$$

Hence the extended action functional for the problem under consideration takes the form

$$A\ \left(u,w\right)=\int_{\Omega}L dx \tag{9}$$

Following denotations introduced in previous Section we recall that $W=\left(C^1(\Omega)\right)^m$, $v_* \in \left(C^1(\Omega)\right)^m$ and $w \in M \equiv v_* + \left(C_0^1(\Omega)\right)^m$. In this case equations (7) lead to quasi-linear PDEs of the form

$$\nabla \cdot \left(A(x,w)\nabla w + C(x,w)\right) - B(x,w)w - d(x,w) = 0 \tag{10}$$

The above equations will be considered under boundary condition $w|_{\partial\Omega} = v_*|_{\partial\Omega}$.

4. Application to problems with constraints

The starting point of considerations in this section are equations of the form

$$F(x,w,\nabla w) = 0, \quad w \in \left(C^1(\Omega)\right)^m, \quad \Omega \subset R^n \tag{11}$$

which will be referred to as primary field equations. Moreover, it is assumed that V is a linear proper subspace of $\left(C^1(\Omega)\right)^m$. The constraints imposed on w are introduced by condition $w \in M \equiv v_* + V$.

In order to derive the primary field equations from the principle of stationary extended action we assume the lagrange function in the form

$$L = F(x,u,\nabla u) \cdot w \tag{12}$$

Hence extended action functional has the form

$$A\ (u,w) = \int_\Omega L dx \tag{13}$$

Using the principle of stationary extended action (5) we have

$$\int_\Omega \left[\frac{\bar{\partial} L}{\partial w} - \nabla \cdot \left(\frac{\bar{\partial} L}{\partial \nabla w}\right)\right] h dx + \nabla \cdot \int_\Omega \frac{\bar{\partial} L}{\partial \nabla w} \cdot h dx = 0 \qquad \forall h \in V \tag{14}$$

Let us observe that $\dfrac{\bar{\partial} L}{\partial \nabla w} = 0$. Hence from equations (14) we obtain

$$\int_\Omega F(x,w,\nabla w) \cdot h dx = 0 \qquad \forall h \in V \tag{15}$$

which lead to primary field equations (11).

If $F, h \in V \equiv H$ - Hilbert space then (15) represent orthogonality conditions approximating primary field equations $F\left(x, w, \nabla w\right) = 0$.

Conclusions

The concept of extended action functional was proposed in this paper. It was shown that this concept leads to principle of stationary extended action (PSEA). The proposed principle makes it possible to derive PDEs with quasi-linear operators. On that account the proposed procedure constitutes new convenient tool for formulation mathematical models. The problems related to some applications of this approach will be studied in forthcoming papers.

References

[1] Gelfand I.M., Fomin S.V., Calculus of variation, trans. by R..A. Silverman, Prentice-Hall, Englewood Cliffs 1963.

[2] Reddy J.N., Applied functional analysis and variational methods in engineering, McGraw-Hill Book Company 1986.