TEMPERATURE DISTRIBUTION IN A CIRCULAR PLATE HEATED BY A MOVING HEAT SOURCE

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Abstract. This paper concerns the heat conduction problem in a circular thin plate subjected to the activity of a heat source which changes its place on the plate surface with time. The heat source moves along an circular trajectory round centre of the plate with constant angular velocity. The solution of the problem is obtained in an analytical form by using the Green's function method.

Introduction

The solutions of heat conduction problems in circular plates with moving heat sources are presented in papers [1-3]. An analytical approach to multi-dimensional heat conduction in composite circular cylinder subjected to generally time-dependent temperature changes has been presented in paper [1]. Boundary temperatures were approximated as Fourier series. The Laplace transform was adopted in deducing the solution of the problem. In paper [2] authors develop an integral transform determineing temperature distribution in a thin circular plate. The plate is subjected to a partially distributed and axisymmetric heat supply. Authors find the temperature field analytically by using the finite Fourier and the finite Hankel transforms. An inverse problem of axially symmetric transient temperature and deflection of a circular plate is solved in paper [3]. A heat flux is assumed on an internal cylindrical surface of the plate. The solution of the problem was obtained by applying the Fourier cosine and the Laplace transforms. A solution to the problem of heat conduction in a rectangular plate subjected to the activity of a moving heat source is presented in paper [4]. The heat source moves along an elliptical trajectory on the plate surface. An exact solution to the problem in an analytical form is obtained by applying the Green's function method.

This paper presents an analytical solution to the heat conduction problem in a circular thin plate which is heated by a moving heat source. The temperature of the plate changes because the heat source moves along circular trajectory on the plate surface. A solution of the heat conduction problem in an analytical form is obtained by using the Green's function method.

1. Problem formulation

Consider a circular thin plate of thickness h and radius $r = b$. This plate is subjected to the activity of a moving heat source. The heat source moves on the plate surface along a circular trajectory at radius r_0 round centre of the plate with constant angular velocity. The temperature distribution $T(r,\phi,z,t)$ of the plate is described by the differential equation of heat conduction [5]:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{k} g(r,\phi,z,t) = \frac{1}{a}\frac{\partial T}{\partial t} \tag{1}$$

where: $T(r,\phi,z,t)$ - temperature, k - thermal conductivity, a - thermal diffusivity, and $g(r,\phi,z,t)$ denotes a volumetric energy generation.

In this study, it is assumed that the thermal energy is provided by the moving heat source (which moves along a circular trajectory on the plate surface). The function $g(r,\phi,z,t)$ occurring in equation (1) has the form

$$g(r,\phi,z,t) = \theta\,\delta(r - r_0)\,\delta(\phi - \varphi(t))\,\delta(z - h) \tag{2}$$

where θ characterises the stream of the heat, $\delta(\ )$ is the Dirac delta function, r_0 is the radius of the circular trajectory along which the heat source moves, $\varphi(t)$ is the function describing the movement of the heat source

$$\varphi(t) = \omega t \tag{3}$$

where ω is angular velocity of the moving heat source.

The differential equation (1) is completed by the following initial and boundary conditions:

$$T(r,\phi,z,0) = 0 \tag{4}$$

$$k\left.\frac{\partial T}{\partial r}\right|_{r=b} = -\alpha_0[T_0 - T(b,\phi,z,t)] \tag{5}$$

$$k\frac{\partial T}{\partial z}(r,\phi,h,t) = \alpha_0[T_0 - T(r,\phi,h,t)] \tag{6}$$

$$k\frac{\partial T}{\partial z}(r,\phi,0,t) = -\alpha_0[T_0 - T(r,\phi,0,t)] \tag{7}$$

where α_0 is the heat transfer coefficient, T_0 is the known temperature of the surrounding medium.

2. Solution of the problem

The solution of the problem in an analytical form is obtained by using the properties of the Green's function (GF). The GF of the heat conduction problem describes the temperature distribution induced by the temporary, local energy impulse. The function is a solution to the following differential equation [5]:

$$\frac{\partial^2 G}{\partial r^2}+\frac{1}{r}\frac{\partial G}{\partial r}+\frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2}+\frac{\partial^2 G}{\partial z^2}-\frac{1}{a}\frac{\partial G}{\partial t}=\frac{\delta(r-\rho)\delta(\phi-\phi')\delta(z-\zeta)\delta(t-\tau)}{r} \quad (8)$$

The Green's function satisfies the initial and homogeneous boundary conditions analogous to the conditions (4)-(7).

The GF for the considered heat conduction problem may be written in the form of a series:

$$G(r,\phi,z,t)=\sum_{m=-\infty}^{\infty} g_m(r,z,t)\cos m(\phi-\phi') \quad (9)$$

Substituting (9) into equation (8) gives

$$\frac{\partial^2 g_m}{\partial r^2}+\frac{1}{r}\frac{\partial g_m}{\partial r}+\frac{\partial^2 g_m}{\partial z^2}-\frac{m^2}{r^2}g_m-\frac{1}{a}\frac{\partial g_m}{\partial t}=\frac{\delta(r-\rho)\,\delta(z-\zeta)\delta(t-\tau)}{2\pi r} \quad (10)$$

The initial and boundary conditions are in the form

$$g_m(r,z,0)=0, \qquad \left.\left(\frac{\partial g_m}{\partial r}-\mu_0\, g_m\right)\right|_{r=b}=0$$

$$\left.\left(\frac{\partial g_m}{\partial z}-\mu_0\, g_m\right)\right|_{z=0}=0, \qquad \left.\left(\frac{\partial g_m}{\partial z}+\mu_0 g_m\right)\right|_{z=h}=0 \quad (11)$$

where $\mu_0=\dfrac{\alpha_0}{k}$.

The solution of the initial-boundary problem (9)-(11) can be presented in the form of a series

$$g_m(r,z,t)=\sum_{n=1}^{\infty}\Gamma_{mn}(r,t)\psi_n(z) \quad (12)$$

where $\Psi_n(z)$ are eigenfunctions of the following boundary problem

$$\frac{\partial^2\psi_n}{\partial z^2}+\beta^2\psi_n(z)=0 \quad (13)$$

$$\left.\left(\frac{d\,\psi_n}{d\,z}-\mu_0\,\psi_n\right)\right|_{z=0}=0\,,\quad \left.\left(\frac{d\,\psi_n}{d\,z}+\mu_0\,\psi_n\right)\right|_{z=h}=0 \tag{14}$$

The functions $\psi_n(z)$ are expressed as [6]

$$\psi_n(z)=\beta_n\cos\beta_n z+\mu_0\sin\beta_n z,\quad n=1,2,... \tag{15}$$

where β_n are roots of the equation

$$2\mu_0\,\beta_n\cos\beta_n h-\left(\beta_n^2-\mu_0{}^2\right)\sin\beta_n h=0 \tag{16}$$

These functions are pairwise orthogonal so that the following condition is satisfied

$$\int_0^h \psi_n(z)\psi_m(z)dz=\begin{cases}0 \text{ for } n\neq m\\ Q_n \text{ for } n=m\end{cases} \tag{17}$$

where $$Q_n=\int_0^h(\psi_n(z))^2dz=\frac{h}{2}\left(\beta_n^2+\mu_0^2\right)\left(1+\frac{\beta_n^2+\mu_0{}^2}{2\mu_0 h\beta_n^2}\sin^2\beta_n h\right) \tag{18}$$

The Dirac function $\delta(z-\zeta)$ may be written in the form

$$\delta(z-\zeta)=\sum_{n=1}^{\infty}\frac{\psi_n(z)\psi_n(\zeta)}{Q_n} \tag{19}$$

Substituting Eqs. (12) and (19) into Eq. (10) gives

$$\frac{\partial^2\Gamma_{mn}}{\partial r^2}+\frac{1}{r}\frac{\partial\Gamma_{mn}}{\partial r}-\left(\beta_n^2+\frac{m^2}{r^2}\right)\Gamma_{mn}-\frac{1}{a}\frac{\partial\Gamma_{mn}}{\partial t}=\frac{\psi_n(\zeta)}{Q_n}\,\frac{\delta(r-\rho)\delta(t-\tau)}{2\pi r} \tag{20}$$

The initial and boundary conditions are

$$\Gamma_{mn}(r,0)=0\,,\qquad \left.\left(\frac{\partial\Gamma_{mn}}{\partial r}-\mu_0\Gamma_{mn}\right)\right|_{r=b}=0 \tag{21}$$

In order to solve the problem (20)-(21), the function $\Gamma_{mn}(r,t)$ is written in the form

$$\Gamma_{mn}(r,t)=\sum_{k=1}^{\infty}R_{mk}(r)T_{mnk}(t) \tag{22}$$

where functions $R_{mk}(r)$ are obtained as solutions of the Bessel's equation

$$\frac{\partial^2 R_{mk}}{\partial r^2}+\frac{1}{r}\frac{\partial R_{mk}}{\partial r}+\left(\gamma_{mk}^2-\frac{m^2}{r^2}\right)R_{mk}(r)=0 \tag{23}$$

where γ_{mk} are separate constants. Moreover, the following conditions are satisfied

$$\lim_{r\to 0}\left|R_{mk}(r)\right| < +\infty\,, \qquad R'_{mk}(b) - \mu_0 R_{mk}(b) = 0 \tag{24}$$

The solution of equation (23) takes the form

$$R_{mk}(r) = C_1 J_m(\gamma_{mk} r) + C_2 Y_m(\gamma_{mk} r) \tag{25}$$

where Jν(·) denotes the Bessel function of the first kind of order ν. Using the first condition (24), $C_2 = 0$ was received. Therefore the solution of Eq. (25) is

$$R_{mk}(r) = C_1 J_m(\gamma_{mk} r) \tag{26}$$

Substituting the functions $R_{mk}(r)$ into the second condition (24) one obtains

$$b\gamma_{mk} J_{m-1}(b\gamma_{mk}) - (m + b\mu_0) J_m(b\gamma_{mk}) = 0 \tag{27}$$

The equation (27) is then solved numerically with respect to the unknown γ_{mk}.
Note that the functions $R_{mk}(r)$ satisfy the orthogonality conditions:

$$\int_0^b r\, J_m(\gamma_{mk} r) J_m(\gamma_{mk'} r)\, dr = 0 \quad \text{for} \quad k' \neq k \tag{28}$$

$$\int_0^b r\, J_m^2(\gamma_{mk} r)\, dr = \frac{b}{2\gamma_{mk}} J_m(\gamma_{mk} b) J_{m+1}(\gamma_{mk} b) \ \text{for} \ \ k = 1, 2, \ldots \tag{29}$$

Hence, taking into account (26) in Eq. (22), the function $\Gamma_{mn}(r,t)$ is obtained in the form

$$\Gamma_{mn}(r,t) = \sum_{k=1}^{\infty} J_m(\gamma_{mk} r) T_{mnk}(t) \tag{30}$$

where $$T_{mnk}(t) = -\frac{a}{\pi b} \frac{\gamma_{mk}\ J_m(\gamma_{mk}\rho)}{J_m(\gamma_{mk} b) J_{m+1}(\gamma_{mk} b)} e^{-\kappa\left(\beta_n^2 + \gamma_{mk}^2\right)(t-\tau)}$$

Finally the Green’s function G has the form

$$G(r,\phi,z,t,\rho,\phi',\varsigma,\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{Q_n} \Gamma_{mn}(r,t) \psi_n(z) \psi_n(\zeta) \cos m(\phi - \phi') \tag{31}$$

The temperature distribution $T(r,\phi,z,t)$ is expressed by the Green’s function G as follows

$$T(r,\phi,z,t) = \int_0^t d\tau \int_0^b d\rho \int_0^{2\pi} d\phi \int_0^h g(\rho,\phi',\varsigma,\tau) G(r,\phi,z,t;\rho,\phi',\varsigma,\tau) dz \tag{32}$$

After evaluation of the integrals in the space domain and using Eq. (2) one obtains the temperature $T(r,\phi,z,t)$ in the form

$$T(r,\phi,z,t) = \Theta\int_0^t G(r,\phi,z,t;r_0,\varphi(\tau\},h,\tau)d\tau \tag{33}$$

Substituting the Green's function (31) into Eq. (33) gives

$$T(r,\phi,z,t) = \frac{\Theta a}{\pi b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A_{mnk} J(\gamma_{mk} r_0) J(\gamma_{mk} r) \Psi_n(z) \Psi_n(h) I_{mnk}(t) \tag{34}$$

where

$$A_{mnk} = \frac{1}{Q_n}\frac{\gamma_{mk}}{J_m(\gamma_{mk} b) J_{m+1}(\gamma_{mk} b)}$$

$$I_{mnk}(t) = \int_0^t \cos m(\phi - \varphi(\tau)) Exp[-\kappa(\beta_n^2 + \gamma_{mn}^2)(t-\tau)d\tau$$

and γ_{mk} are roots of the equation (27). The integrals can be evaluated through a series expansion [4] or numerically.

Conclusions

In this paper, an analytical model to describe the three-dimensional temperature field for a circular plate with a heat source which moves over its surface was established. The moving heat source causes cyclic heating of various plate areas. The temperature distribution in the considered plate in an analytical form was obtained using the time-dependent Green's function.

References

[1] Lu X., Tervola P., Viljanen M., Transient analytical solution to heat conduction in composite circular cylinder, International Journal Heat and Mass Transfer 2006, 49, 341-348.

[2] Khobragade N.L., Deshmukh K.C., Thermal deformation in a thin circular plate due to a partially distributed heat supply, Sadhana 2005, 30, 555-563.

[3] Khobragade N.L., Deshmukh K.C., An inverse quasi-static thermal deflection problem for a thin clamped circular plate, Journal of Thermal Stresses 2005, 28, 353-361.

[4] Kidawa-Kukla J., Temperature distribution in a rectangular plate heated by moving heat source, International Journal Heat and Mass Transfer 2008, 51, 865-872.

[5] Beck J.V., et al, Heat Conduction Using Green's Functions, Hemisphere Publishing Corporation, London 1992.

[6] Duffy D.G., Green's Functions with Applications - Studies in Advanced Mathematics, Boca Raton, London, New York 2001.