THE FINITE DIFFERENCE METHOD FOR FRACTIONAL CATTANEO-VERNOTTE EQUATION

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Abstract. In this work a numerical solution of modified Cattaneo-Vernotte equation is presented. This equation is obtained by replacing the second order time derivative by the fractional derivative in Caputo sense. In order to solve the problem with classical boundary-initial conditions, the finite difference method is applied. In the final part of the paper the examples of computations are shown.

1. Formulation of the problem

Fourier's law of heat conduction specifies that the heat flux is directly proportional to the temperature gradient at any time and at any point in the medium. Fourier's law implies an infinite speed of thermal signal propagation. Using the concept of a finite heat propagation velocity, Cattaneo and Vernotte formulated a modified unsteady heat conduction equation [1] (here the 1D problem is considered)

$$q(x,t+\tau) = -\lambda \frac{\partial T(x,t)}{\partial x} \tag{1}$$

where q is is the heat flux, T is the temperature, λ is the thermal conductivity, t is the time, x is the spatial coordinate and τ is a thermodynamic property of materials called the thermal relaxation time which represents the time necessary for the initiation of the heat flux after a temperature gradient has been imposed. Equation (1) proves that the heat flux does not start immediately, but rather grows gradually, depending on the thermal relaxation time, after the application of the temperature gradient. Equation (1) reduces to the classical Fourier law for $\tau = 0$.

The heat flux $q(x,\ t+\tau)$ can be expanded in generalized Taylor series of fractional order [2] as

$$q(x,t+\tau) = \sum_{k=0}^{\infty} \frac{\tau^{\alpha k}}{\Gamma(\alpha k+1)} \frac{{}^{C}\partial^{(\alpha k)} q(x,t)}{\partial t^{(\alpha k)}}, \quad \alpha \in (0,1] \tag{2}$$

where

$$\frac{{}^{C}\partial^{(\beta)}q(x,t)}{\partial t^{(\beta)}} := {}_{0}^{C}D_{t}^{\beta}q(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)}\displaystyle\int_{0}^{t}\frac{\frac{\partial^{m}}{\partial\xi^{m}}q(x,\xi)}{(t-\xi)^{\beta-m+1}}d\xi & \text{for } \beta\notin\mathbf{N} \\ \dfrac{\partial^{m}}{\partial\xi^{m}}q(x,\xi) & \text{for } \beta\in\mathbf{N} \end{cases} \tag{3}$$

where $m \in \mathbf{N}$, $m - 1 < \beta \le m$. The operator (3) is the fractional derivative defined in Caputo sense [3-5]. Taking into account only two terms of the generalized Taylor series (2) and introducing into (1), the following unsteady heat conduction equation is obtained

$$q(x,t) + \frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\frac{{}^{C}\partial^{(\alpha)}q(x,t)}{\partial t^{(\alpha)}} = -\lambda\frac{\partial T(x,t)}{\partial x} \tag{4}$$

The energy conservation equation is given as

$$c\rho\frac{\partial T(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x} \tag{5}$$

where ρ is the density and c is the specific heat of the medium. Putting (4) into (5) one obtains the fractional Cattaneo-Vernotte (C-V) equation

$$\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\frac{{}^{C}\partial^{(\alpha+1)}T(x,t)}{\partial t^{(\alpha+1)}} + \frac{\partial T(x,t)}{\partial t} = a\frac{\partial^{2}T(x,t)}{\partial x^{2}} \tag{6}$$

where $a = \lambda/(c\rho)$. By setting $\alpha = 1$, the equation (6) reduces to the classical C-V equation [1, 6, 7]

$$\tau\frac{{}^{C}\partial^{2}T(x,t)}{\partial t^{2}} + \frac{\partial T(x,t)}{\partial t} = a\frac{\partial^{2}T(x,t)}{\partial x^{2}} \tag{7}$$

It should be pointed out that for $\tau = 0$ the equations (6) and (7) reduce to the Fourier heat transfer equation.

The equations (6) and (7) are supplemented by the boundary conditions

$$\begin{aligned} x = 0: &\quad T(x,t) = g_{0} \\ x = L: &\quad T(x,t) = g_{L} \end{aligned} \tag{8}$$

and initial ones

$$t = 0: \quad T(x,t)\big|_{t=0} = p_{0}, \quad \left.\frac{\partial T(x,t)}{\partial t}\right|_{t=0} = p_{1} \tag{9}$$

2. Numerical method

In order to develop a discrete form of equation (6), two homogenous grids are introduced - spatial: $0 = x_0 < x_1 < x_2 < \ldots < x_i < x_{i+1} < \ldots < x_N = L$ with the mesh step $\Delta x = x_{i+1} - x_i$ and temporal: $0 = t_0 < t_1 < t_2 < \ldots < t_f < t_{f+1} < \ldots < t_F$ with the time step $\Delta t = t_{f+1} - t_f$. A value of the function T at the point x_k for the moment of time t_f is denoted as $T_k^f = T(x_k, t_f)$.

The discretization method of the fractional operator was described in detail in [3]. According to fractional calculus [4, 5] the following relation occurs (here only valid for $1 < \alpha + 1 \le 2$)

$$\frac{{}^{C}\partial^{(\alpha+1)} T(x,t)}{\partial t^{(\alpha+1)}} = \frac{{}^{GL}\partial^{(\alpha+1)}}{\partial t^{(\alpha+1)}} \left(T(x,t) - T(x,t)\big|_{t=0} - t \left. \frac{\partial T(x,t)}{\partial t} \right|_{t=0} \right) \tag{10}$$

where

$$\frac{{}^{GL}\partial^{(\alpha+1)} T(x,t)}{\partial t^{(\alpha+1)}} = \lim_{\Delta t \to 0} (\Delta t)^{-(\alpha+1)} \sum_{k=0}^{[t/\Delta t]} (-1)^k \binom{\alpha+1}{k} T(x, t - k\Delta t) \tag{11}$$

is the definition of the fractional derivative in Grűnwald-Letnikov sense. Using formula (10) one can write discrete form of the fractional derivative in Caputo sense

$$\left. \frac{{}^{C}\partial^{(\alpha+1)} T(x,t)}{\partial t^{(\alpha+1)}} \right|_{x=x_i}^{t=t_f} \approx (\Delta t)^{-(\alpha+1)} \sum_{k=0}^{f} u_k^{(\alpha+1)} \left(T_i^{f-k} - p_0 - f\, \Delta t\, p_1 \right) \tag{12}$$

where

$$u_k^{(\alpha+1)} = (-1)^k \binom{\alpha+1}{k} = (-1)^k \frac{\Gamma(\alpha+2)}{\Gamma(k+1)\Gamma(\alpha-k+2)} \tag{13}$$

or using the recurrence relationship [4], one can compute the coefficients in a simple way

$$u_0^{(\alpha+1)} = 1, \quad u_k^{(\alpha+1)} = \left(1 - \frac{2+\alpha}{k} \right) u_{k-1}^{(\alpha+1)} \quad \text{for } k = 1, 2, \ldots \tag{14}$$

The classical finite difference approximations for numerical differentiation of the first temporal and second spatial derivatives occurring in equation (6) are assumed [8, 9]. Introducing the discrete forms of derivatives into (6), a finite difference scheme depending on the weight factor σ is obtained (the method is explicit for $\sigma = 1$, partially implicit for $0 < \sigma < 1$ and fully implicit for $\sigma = 0$)

$$
\begin{gathered}
\frac{\tau^{\alpha-1}}{\Gamma(\alpha+1)}(\Delta t)^{-(\alpha+1)}\sum_{k=0}^{f+1}u_k^{(\alpha+1)}\left(T_i^{f+1-k}-p_0+(f+1)\Delta t\,p_1\right)+\frac{T_i^{f+1}-T_i^f}{\Delta t} \\
=a\frac{\sigma\left(T_{i-1}^f-2T_i^f+T_{i+1}^f\right)+(1-\sigma)\left(T_{i-1}^{f+1}-2T_i^{f+1}+T_{i+1}^{f+1}\right)}{(\Delta x)^2} \quad \text{for } i=1,\dots,N-1
\end{gathered} \tag{15}
$$

and for boundary nodes

$$
T_0^{f+1}=g_0,\;\; T_N^{f+1}=g_L \tag{16}
$$

The above scheme described by expressions (15) and (16) can be written in a matrix form as

$$
\mathbf{A}\cdot\mathbf{T}^{f+1}=\mathbf{B} \tag{17}
$$

where

$$
\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 0 \\ a_{-1} & a_0 & a_1 & 0 & \dots & 0 \\ 0 & a_{-1} & a_0 & a_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \ddots & \vdots \\ 0 & \dots & 0 & a_{-1} & a_0 & a_1 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix},\quad
\mathbf{B}=\begin{bmatrix} g_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N-1} \\ g_N \end{bmatrix},\quad
\mathbf{T}^{f+1}=\begin{bmatrix} T_0^{f+1} \\ T_1^{f+1} \\ T_2^{f+1} \\ \vdots \\ T_{N-1}^{f+1} \\ T_N^{f+1} \end{bmatrix} \tag{18}
$$

with

$$
\begin{gathered}
a_0=\frac{\tau^\alpha}{\Gamma(\alpha+1)(\Delta t)^{\alpha+1}}+\frac{1}{\Delta t}+\frac{2a(1-\sigma)}{(\Delta x)^2} \\
a_{-1}=a_1=\frac{a(\sigma-1)}{(\Delta x)^2} \\
b_i=\frac{a\sigma}{(\Delta x)^2}\left(T_{i-1}^f-2T_i^f+T_{i+1}^f\right)+\frac{1}{\Delta t}T_i^f-\frac{\tau^\alpha}{\Gamma(\alpha+1)(\Delta t)^{\alpha+1}}\sum_{k=1}^{f+1}u_k^{(\alpha+1)}T_i^{f+1-k} \\
+\frac{\tau^\alpha}{\Gamma(\alpha+1)(\Delta t)^{\alpha+1}}\sum_{k=0}^{f+1}u_k^{(\alpha+1)}\left(p_0+(f+1)\Delta t\,p_1\right)
\end{gathered} \tag{19}
$$

and $\boldsymbol{T}^{f+1}$ is the vector of unknown values of the function T.

It is observed that initial conditions influence on the all values of function at every computational time step. In opposite to the classical derivatives, which are approximated locally, the characteristic feature of time fractional derivatives is the dependence on values of all previous time levels.

3. Results of computations

In this section the results of calculations are presented. In all presented simulations the following parameters have been assumed: L = 0.01 m, p_0 = 37°C, p_1 = = 0°C/s, g_0 = 0°C, g_L = 0°C, $a = \lambda/(c\rho) = 2.67\cdot10^{-7}$ m^2/s. The equation (6) has been solved using the finite difference method with mesh step Δx = 0.00002m and time step Δt = 0.01s. The computations have been done for relaxation time τ = 2s. For comparison of the results of numerical solution of the fractional C-V equation (7), the analytical solution of the classical C-V equation (6) [7, 10] is calculated.

In Figure 1 the analytical solution of equation (7) and the numerical solution of equation (6) are presented. The lines are drawn at the same moment of time. Figure 2 illustrates the courses of temperature at two points: x = 2 mm and x = 5 mm.

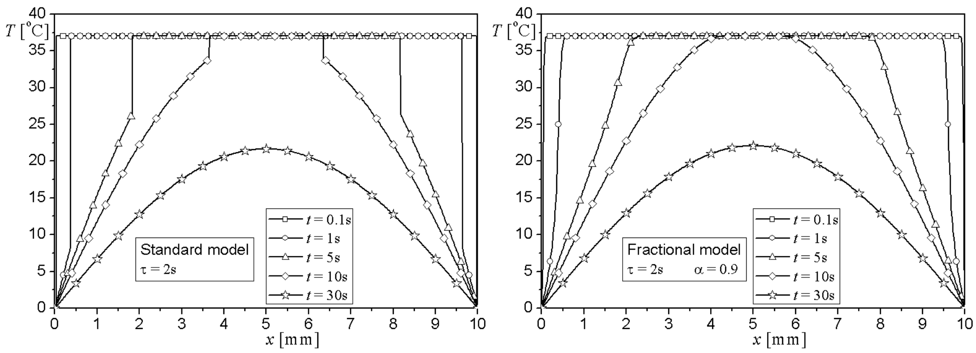

Fig. 1. Comparison of analytical solution of equation (7) over space for τ = 2 s (left-side) with numerical solution of equation (6) over space for τ = 2 s and α = 0.9 (right-side)

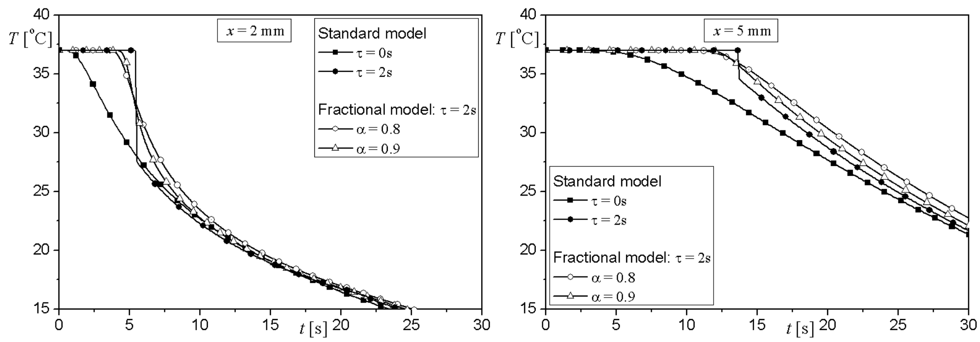

Fig. 2. Solutions of equations (6) and (7) over time in points: x = 2 mm (left-side) and x = 5 mm (right-side) - comparison of both models

Conclusions

Summing up, the finite difference method has been applied to solve modified C-V equation. In the solution of the classical C-V equation one can observe that sharp front has been occurred in profile of temperature. Analyzing the solution of the fractional C-V equation, smoother transition of temperature front for smaller values of α has been observed. It should be pointed out that a time lag in initial phase has been occurred in both solutions.

The computations of numerical solution of fractional differential equations have been generally limited to short time of simulation. By the definition of the fractional derivative, the solution of equations at current time level depends on the solutions at the all previous time levels. All previous solutions must be available for computations at current time level.

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