CONVERGENCE OF AN ASYNCHRONOUS ITERATIVE ALGORITHM

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Abstract. The generalized model of asynchronous iterations and its application for implementation of the algebraic iterative algorithm ART on a non-synchronous computer structure is presented. The sufficient conditions for convergence of this algorithm are considered.

Introduction

One of the most perspective areas of parallel computations is an elaboration of asynchronous realizations of iterative algorithms. The main characteristic of the organization of asynchronous computations is that the solution is obtained during a non-synchronous interaction of processor elements of a parallel structure. Each processor of this PCS updates the values of corresponding components of the solution using available information about other components of the solution, and it obtains this information from local processors or shared memory without waiting their full update. The researches showed that the asynchronous realizations of parallel algorithms are more efficient from the point of view of their speed of convergence in many important cases [1-3]. Note that the convergence of asynchronous algorithms and their synchronous prototypes may be different. Some models of asynchronous iterative methods for image reconstruction are considered in [4-8]. In section 1 it is given some generalization of these models (see also [9-11]).

Consider a system of linear algebraic equations:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{p} \tag{1}$$

where:

- $\mathbf{A} = (a_{ij}) \in \mathbf{R}^{m,n}$ is the matrix of coefficients
- $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$
- $\mathbf{p} = (p_1, p_2, \ldots, p_m)^T \in \mathbf{R}^m$

which has a few characteristics: it is a rectangular, it has a very large dimension and $\mathbf{A}$ is a sparse matrix.

For solving this system it is often used different kinds of algebraic iterative algorithms the most well-known of which are the additive algorithm ART [12-15]. These algorithms are very flexible and allow to apply different *a priory* information about object before its reconstruction that is especially very important when we have an incomplete projection data. Denote

$$\mathbf{P}_i(\mathbf{x}) = \mathbf{x} - \frac{(\mathbf{a}^i, \mathbf{x}) - p_i}{\|\mathbf{a}^i\|^2}\mathbf{a}^i \tag{2}$$

$$\mathbf{P}_i^{\lambda} = (1-\lambda)\mathbf{I} + \lambda \mathbf{P}_i \tag{3}$$

where $\mathbf{a}^i$ is the i-th row of the matrix **A**, and λ is a relaxation parameter.

Algorithm 1 (ART-1)

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$\mathbf{x}^{(k+1)} = \mathbf{C}\mathbf{P}_i^{\lambda_k}\mathbf{x}^{(k)} \quad (i = 1, 2, ..., m) \tag{4}$$

where $\mathbf{P}_i^{\lambda_k}$ are operators defined by (3), λ_k are relaxation parameters, **C** is a constraining operator, and $i(k) = k(\bmod m) + 1$.

Algorithm 1 was proposed by Kaczmarz [16] and independently discovered and investigated by Herman, Lent, Rowland in [14]. It was used successfully in application of computerized tomography in medicine. This algorithm runs through all equations cyclically with modification of the present estimate $\mathbf{x}^{(k)}$ in such a way that the present equation with index i is fulfilled.

In section 2 the general model of asynchronous iterations is applied for implementation of the algebraic iterative algorithm ART-1 and there are given the sufficient conditions for convergence of this algorithm.

1. Basic notions and concepts

First chaotic and asynchronous algorithms for image reconstruction were proposed and studied in [6, 7]. These algorithms are based on the methods of asynchronous iterations introduced first by Chasan and Miranker [5]. The further development of these methods and their generalizations for the case of non-linear operators was obtained by Baudet [4].

Recall some important notions of the theory of chaotic and asynchronous iterations [2, 4 , 5, 17, 18].

Definition 1

A sequence of nonempty subsets $I = \{I_k\}_{k=0}^{\infty}$ of the set $\{1,2\ldots,m\}$ is **a sequence of chaotic sets** if

$$\limsup_{k\to\infty} I_k = \{1,2,...,m\} \tag{5}$$

(another words, if each integer $j \in \{1,2,...,m\}$ appears in this sequence an infinite number of times).

For the first time such sequences were used by Baudet [4].

Definition 2

If any subset of a sequence of chaotic sets I has the form $I_k = \{j_k\}_{k=0}^{\infty}$, where $j_k \in \{1,2,...,m\}$ (i.e. each set consists of only one element), then the sequence I is called **acceptable** (or **admissible**).

Suppose that PCS (Parallel Computing System) consists of m processors working local independently. In this case the notion of the sequence of chaotic sets has a simple interpretation: it sets the time diagram of work of each processor during non-synchronous work of PCS. So the subset I_k is the set of the numbers of those processors which access the central processor at the same time.

Note, that the definition of the sequence of chaotic sets can be given in the following equivalent form:

Lemma 1

Let $I = \{I_k\}_{k=0}^{\infty}$ be a sequence of nonempty subsets of the set $\{1,2...,m\}$. Then the following conditions are equivalent:

1) $\limsup_{k\to\infty} I_k = \{1,2,...,m\}$
2) the set $\{ k \mid i \in I_k \}$ is unlimited for each $i = 1,2...,m$
3) for each $j \in \mathbf{N}$ there exists $p(j) \in \mathbf{N}$ such that the following condition satisfies:

$$\bigcup_{i=j+1}^{j+p(j)} I_i = \{1,2,...,m\} \tag{6}$$

For any sequence of chaotic sets the numbers $p(j)$ depends on a number j. In practice and for researching the convergence of asynchronous implementations of iterative processes there are more important sequences of chaotic sets, for which these numbers do not depend on number j.

Definition 3

If for a sequence of chaotic sets $I = \{I_k\}_{k=0}^{\infty}$ the numbers $p(j)$ defined by (6) do not depend on the choice of number j, i.e., $p(j) = T = \text{const}$ for each $j \in \mathbf{N}$, then this sequence is called **regular**, and the number T is called the **number of regularity** of the sequence I.

Note, that this definition coincides with a concept of a regular sequence, introduced in [19] for the case of admissible sequence. In this work El Tarazi obtained important results introducing the obvious model for the class of asynchronous algorithms and giving the first correct conditions of convergence in the non-linear case of contraining operators.

Definition 4

A sequence $J = \{\sigma(k)\}_{k=1}^{\infty}$ of m-dimensional vectors $\sigma(k) = \{\sigma_1(k), \sigma_2(k), ..., \sigma_m(k)\}$ with integer coordinates, satisfying the following conditions:

1) $0 \le \sigma_i(k) \le k-1$ (7)

2) $\lim\limits_{k\to\infty} \sigma_i(k) = \infty$ (8)

for each $i = 1,2...,m$ and $k \in \mathbf{N}$, is called a **sequence of delays**.

In the case, when instead of condition 2) it holds the following condition:
2') there exists a fixed number $L \in \mathbf{N}$ such that

$$k - \sigma_i(k) \le L \tag{9}$$

for each $k \in \mathbf{N}$ and $i = 1,2...,m$, the sequence is called a **sequence with limited delays** and the number L is called a **delay**, or an **asynchronous measure**.

The sequence of delays determines the numbers of using iterations by each fixed processor, and the number L shows a depth of used iterations and actually reflects possibilities of the concrete computing system. For synchronous implementation of the iterative process the difference $k - \sigma_i(k)$ is equal to 0 for $\forall i =$ $= 1,2...,m$ and $k \in \mathbf{N}$.

Consider the definition of some generalized model of asynchronous computational process (see [9, 10, 20]).

Definition 5

Let there exist a set of nonlinear operators $\mathbf{T}_i$: $\mathbf{R}^n \to \mathbf{R}^n$, $i \in \{1,2,...,m\}$ and an initial value $\mathbf{x}^{(0)} \in \mathbf{R}^n$. A **generalized model of the asynchronous iterations with**

limited delays for the set of operators $\mathbf{T}_i$, $i = 1,2,...,m$ is called a method of building the sequence of vectors $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$, which is given recursively by the following scheme:

$$\begin{aligned} \mathbf{y}^{k,i} &= \begin{cases} \mathbf{T}_i \mathbf{x}^{(\sigma^i(k))}, & \textit{if } i \in I_k \\ \mathbf{y}^{k-1,i}, & \textit{otherwise} \end{cases} \\ \mathbf{x}^{(k)} &= \mathbf{S}\left(\mathbf{x}^{(k-1)}, \{\mathbf{y}^{k,i}\}_{i \in I_k}\right), \end{aligned} \tag{10}$$

where $I = \{I_k\}_{k=1}^{\infty}$ is a sequence of chaotic sets such that $I_k \subset \{1,2,...,m\}$ and $J_i = \{\sigma^i(k)\}_{k=1}^{\infty}$ are sequences of limited delays ($i = 1,2,...,m$).

2. Convergent conditions of an asynchronous iterative algorithm

Apply the generalized model of asynchronous iterations for an implementation of the algorithm ART-1 on a non-synchronous computer structure. In this case there results the following asynchronous algorithm, where the numbers of operators are chosen by a chaotic way.

Algorithm 2 (ASIRT)

1. $\mathbf{x}^{(0)} \in \mathbf{R}^n$ is an arbitrary vector;
2. The k+1-th iteration is calculated in accordance with the following scheme:

$$\begin{aligned} \mathbf{y}^{k+1,i} &= \begin{cases} \mathbf{P}_i^{\lambda_k} \mathbf{x}^{(\sigma^i(k))}, & \textit{if } i \in I_k \\ \mathbf{y}^{k,i}, & \textit{otherwise} \end{cases} \\ \mathbf{x}^{(k+1)} &= \sum_{i \in I_k} \gamma_i^k \mathbf{y}^{k+1,i}, \quad (i = 1,2,...,m), \end{aligned} \tag{11}$$

where $\mathbf{P}_i^{\lambda_k}$ are operators defined by (2), (3); λ_k are relaxation parameters; γ_i^k are positive real numbers for each $k \in \mathbf{N}$; $I = \{I_k\}_{k=1}^{\infty}$ is a sequence of chaotic sets such that $I_k \subset \{1,2,...,m\}$; $J_i = \{\sigma^i(k)\}_{k=1}^{\infty}$ are sequences of delays.

In this section there will be proved the sufficient conditions for the convergence of this algorithm, which are given by the following theorem:

Theorem 1

Let system (1) be consistent, $I = \{I_k\}_{k=1}^{\infty}$ be a regular sequence of chaotic sets $I_k \subset \{1,2,...,m\}$ with a number of regularity T, $\{\sigma^i(k)\}_{k=1}^{\infty}$ be sequences with limited delays and $\sigma_j^i(k) = \sigma_i(k)$, and let a delay be equal to T. If $0 < \lambda_k < 2$, γ_i^k are positive real numbers with property $\sum_{i \in I_k} \gamma_k^i = 1$, then for every point $\mathbf{x}^{(0)} \in \mathbf{R}^n$ the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ defined by algorithm 2 converges to some point $\mathbf{x}^* \in H$, which is a fixed point of orthogonal projection operators $\mathbf{P}_i$ (i = 1,2,...,m).

Definition 6

A continuous operator $\mathbf{Q}$ belongs to the set F_1 of operators from $\mathbf{R}^n$ into $\mathbf{R}^n$ if it satisfies the following conditions:

(1) $\|\mathbf{Qx} - \mathbf{Qy}\| \le \|\mathbf{x} - \mathbf{y}\|$ for $\forall\ \mathbf{x},\mathbf{y} \in \mathbf{R}^n$.

(2) if $\|\mathbf{Qx} - \mathbf{Qy}\| = \|\mathbf{x} - \mathbf{y}\|$ then

(a) $\mathbf{Qx} - \mathbf{Qy} = \mathbf{x} - \mathbf{y}$

(b) $(\mathbf{x} - \mathbf{y}, \mathbf{Qy} - \mathbf{y}) = 0$.

The notion of such kind of operators was first introduced and studied in the work [21]. These operators are very important in the theory of convex sets.

Lemma 2

If $0 < \lambda < 2$, then an operator $\mathbf{P}^\lambda \in F_1$ for any projection $\mathbf{P}$.

Proof.

1. Let $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, then
$\|\mathbf{P}^\lambda \mathbf{x} - \mathbf{P}^\lambda \mathbf{y}\|^2 = \|(1-\lambda)(\mathbf{x} - \mathbf{y}) + \lambda(\mathbf{Px} - \mathbf{Py})\|^2 = (1 - \lambda)\,\|\mathbf{x} - \mathbf{y}\|^2 +$
$+ 2\lambda(1 - \lambda)\,(\mathbf{x} - \mathbf{y}, \mathbf{Px} - \mathbf{y}\}) + \lambda^2\,\|\mathbf{Px} - \mathbf{Py}\|^2 = \|\mathbf{x} - \mathbf{y}\|^2 + \lambda(\lambda - 2)\,\|\mathbf{x} - \mathbf{y}\|^2 +$
$2\lambda(1 - \lambda)\,(\mathbf{x} - \mathbf{y}, \mathbf{Px} - \mathbf{Py}) + \lambda^2\,\|\mathbf{Px} - \mathbf{Py}\|^2 = \|\mathbf{x} - \mathbf{y}\|^2 + \lambda\mathbf{Q}(\mathbf{x}, \mathbf{y}, \lambda)$,
where
$\mathbf{Q}(\mathbf{x}, \mathbf{y}, \lambda) = \lambda\,\|\mathbf{x} - \mathbf{y}\|^2 - 2\lambda\,(\mathbf{x} - \mathbf{y}, \mathbf{Px} - \mathbf{Py}) + \lambda\,\|\mathbf{Px} - \mathbf{Py}\|^2 - 2\,\|\mathbf{x} - \mathbf{y}\|^2$
$- 2\,(\mathbf{x} - \mathbf{y}, \mathbf{Px} - \mathbf{Py}) = \lambda\,\|\mathbf{x} - \mathbf{y} - \mathbf{Px} + \mathbf{Py}\|^2 - 2\,\|\mathbf{x} - \mathbf{y}\|^2 + 2\,(\mathbf{x} - \mathbf{y}, \mathbf{Px} - \mathbf{Py})$.
Since $\lambda < 2$, there results
$\mathbf{Q}(\mathbf{x}, \mathbf{y}, \lambda) \le 2\,\|\mathbf{x} - \mathbf{y} - \mathbf{Px} + \mathbf{Py}\|^2 - 2\,\|\mathbf{x} - \mathbf{y}\|^2 + 2\,(\mathbf{x} - \mathbf{y}, \mathbf{Px} - \mathbf{Py})$
$= 2\,(\mathbf{Px} - \mathbf{Py}, \mathbf{Px} - \mathbf{Py}) - 2(\mathbf{x} - \mathbf{y}, \mathbf{Px} - \mathbf{Py}) = 2(\mathbf{Px} - \mathbf{Py} - \mathbf{x} + \mathbf{y}, \mathbf{Px} - \mathbf{Py})$
$= [(\mathbf{x} - \mathbf{Px}, \mathbf{Px} - \mathbf{Py}) + (\mathbf{y} - \mathbf{Py}, \mathbf{Px} - \mathbf{Py})] \le 0$.
Taking into account that $\lambda > 0$ there results $\|\mathbf{P}^\lambda \mathbf{x} - \mathbf{P}^\lambda \mathbf{y}\|^2 \le \|\mathbf{x} - \mathbf{y}\|^2$, whence it follows that
$$\|\mathbf{P}^\lambda \mathbf{x} - \mathbf{P}^\lambda \mathbf{y}\| \le \|\mathbf{x} - \mathbf{y}\|$$
2. If $\|\mathbf{P}^\lambda \mathbf{x} - \mathbf{P}^\lambda \mathbf{y}\| = \|\mathbf{x} - \mathbf{y}\|$ then there are only equalities in the chain of previous inequalities. In particular, there results the following equalities:
$$\|\mathbf{x} - \mathbf{y} - \mathbf{Px} + \mathbf{Py}\| = \|\mathbf{x} - \mathbf{y}\|,$$
$$(\mathbf{x} - \mathbf{y}, \mathbf{Py} - \mathbf{y}) = 0$$

whence $\mathbf{Px} - \mathbf{Py} = \mathbf{x} - \mathbf{y}$. Then from the definition of the relaxed projection operator (3) there results

$$\mathbf{P}^{\lambda}\mathbf{x} - \mathbf{P}^{\lambda}\mathbf{y} = (1-\lambda)(\mathbf{x}-\mathbf{y}) + \lambda(\mathbf{Px}-\mathbf{Py}) = \mathbf{x}-\mathbf{y}$$

and

$$(\mathbf{x}-\mathbf{y}, \mathbf{P}^{\lambda}\mathbf{y}-\mathbf{y}) = (\mathbf{x}-\mathbf{y}, (1-\lambda)\mathbf{y} + \lambda\mathbf{Py}-\mathbf{y}) = (\mathbf{x}-\mathbf{y}, \lambda\mathbf{Py}-\mathbf{y}) = \\ = \lambda(\mathbf{x}-\mathbf{y}, \mathbf{Py}-\mathbf{y}) = 0$$

Proof of theorem 1

Suppose that the system of equations (1) is consistent, i.e. there is if only one common point of all operators $\mathbf{P}_i$ ($i = 1,2,...,m$). Show that in this case there is a limit of the sequence defined by algorithm 2 and this limited point is also a common point of all operators $\mathbf{P}_i$ if $0 < \lambda < 2$.

Denote by $\mathbf{T}_k = \sum_{i\in I_k} \gamma_i^k \mathbf{P}_i^{\lambda}$ and show that the operator $\mathbf{T}_k \in F_1$ for $\forall\ k \in \mathbf{N}$. For any $\mathbf{x}$, $\mathbf{y}$

$$\|\mathbf{T}_k\mathbf{x} - \mathbf{T}_k\mathbf{y}\| = \|\sum_{i\in I_k}\gamma_i^k\mathbf{P}_i^{\lambda}\mathbf{x} - \sum_{i\in I_k}\gamma_i^k\mathbf{P}_i^{\lambda}\mathbf{y}\| = \|\sum_{i\in I_k}\gamma_i^k(\mathbf{P}_i^{\lambda}\mathbf{x} - \mathbf{P}_i^{\lambda}\mathbf{y})\| \le \\ \le \sum_{i\in I_k}\gamma_i^k\|\mathbf{P}_i^{\lambda}\mathbf{x} - \mathbf{P}_i^{\lambda}\mathbf{y}\| \le \sum_{i\in I_k}\gamma_i^k\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\|$$

If now $\|\mathbf{T}_k\mathbf{x} - \mathbf{T}_k\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\|$, then from the previous chain of inequalities there results $\|\mathbf{P}_i^{\lambda}\mathbf{x} - \mathbf{P}_i^{\lambda}\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}\|$, whence from lemma 1 it follows that $\mathbf{P}_i^{\lambda}\mathbf{x} - \mathbf{P}_i^{\lambda}\mathbf{y} = \mathbf{x}-\mathbf{y}$, therefore

$$\mathbf{T}_k\mathbf{x} - \mathbf{T}_k\mathbf{y} = \sum_{i\in I_k}\gamma_i^k(\mathbf{P}_i^{\lambda}\mathbf{x} - \mathbf{P}_i^{\lambda}\mathbf{y}) = \sum_{i\in I_k}(\mathbf{x}-\mathbf{y}) = \mathbf{x}-\mathbf{y}$$

Let $\mathbf{z}$ be a common point of all projection operators $\mathbf{P}_i$, i.e. $\mathbf{P}_j\mathbf{z} = \mathbf{z}$ for all $j = 1,2,...,m$. Then

$$\mathbf{T}_k\mathbf{z} = \sum_{i\in I_k}\mathbf{P}_i^{\lambda}\mathbf{z} = \mathbf{z}$$

Therefore taking into account that $\mathbf{T}_k \in F_1$ there results

$$\|\mathbf{x}^{(k)} - \mathbf{z}\| = \|\mathbf{T}_k\mathbf{x}^{(k-1)} - \mathbf{T}_k\mathbf{z}\| \le \|\mathbf{x}^{(k-1)} - \mathbf{z}\| \tag{12}$$

Repeating this process we obtain a sequence of inequalities:

$$\|\mathbf{x}^{(k)} - \mathbf{z}\| \le \|\mathbf{x}^{(k-1)} - \mathbf{z}\| \le ... \le \|\mathbf{x}^{(0)} - \mathbf{z}\|$$

whence it follows that the sequence $\{\|\mathbf{x}^{(k)}-\mathbf{z}\|\}_{k=0}^{\infty}$ is non-increasing and bounded and so it has a limit. Let

$$\alpha=\lim_{k\to\infty}\|\mathbf{x}^{(k)}-\mathbf{z}\| \tag{13}$$

Since $\|\mathbf{x}^{(k)}\| = \|\mathbf{x}^{(k)} - \mathbf{z} + \mathbf{z}\| \le \|\mathbf{x}^{(k)} - \mathbf{z}\| + \|\mathbf{z}\| \le \|\mathbf{x}^{(0)} - \mathbf{z}\| + \|\mathbf{z}\|$, then the sequence $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$ is bounded and hence there exists its convergent subsequence $\{\mathbf{x}^{(p_k)}\}_{k=0}^{\infty}$. Let

$$\lim_{k\to\infty}\mathbf{x}^{(p_k)}=\mathbf{x}^* \tag{14}$$

Consider the set X = (X_1, X_2,...,X_r) of all subsets of the set (1, 2,...,m). Since $\{I_k\}$ is a sequence of chaotic sets, there is if only one subset X_q and an infinite subsequence $\{q_k\}\subset\{p_k\}$ such that $I_{q_k} = X_q$. If there are a few such sequences then we choose an arbitrary of them with the largest number of elements. Then since $\{\mathbf{x}^{(q_k)}\}\subset\{\mathbf{x}^{(p_k)}\}$, from (14) it follows that

$$\lim_{k\to\infty}\mathbf{x}^{(q_k)}=\mathbf{x}^*$$

Therefore taking into account (13) there results the equality

$$\alpha=\lim_{k\to\infty}\|\mathbf{x}^{(q_k)}-\mathbf{z}\|=\|\mathbf{x}^*-\mathbf{z}\| \tag{15}$$

On the other hand since

$$\mathbf{x}^{(q_k)}=\sum_{i\in X_q}\gamma_i^q\mathbf{P}_i^{\lambda}\mathbf{x}^{(q_k-1)}$$

and $\mathbf{z}$ is a common point of all operators $\mathbf{P}_i$, one can build the following sequence of inequalities

$$\|\mathbf{x}^{(q_k)}-\mathbf{z}\|=\|\sum_{i\in X_q}\gamma_i^q\mathbf{P}_i^{\lambda}\mathbf{x}^{(q_k-1)}-\mathbf{z}\|\le\|\mathbf{x}^{(q_k-1)}-\mathbf{z}\|\le\ldots\le\|\mathbf{x}^{(p_l)}-\mathbf{z}\|$$

for some l, whence it follows that

$$\alpha=\|\sum_{i\in X_q}\gamma_i^q\mathbf{P}_i^{\lambda}\mathbf{x}^*-\mathbf{z}\|=\|\mathbf{T}_q\mathbf{x}^*-\mathbf{z}\|$$

Comparing this equality with (15) there results

$$\|\mathbf{T}_q\mathbf{x}^*-\mathbf{z}\|=\|\mathbf{x}^*-\mathbf{z}\|$$

whence $\mathbf{P}_i^\lambda\,\mathbf{x}^* = \mathbf{x}^*$ for $\forall\ i \in X_q$, i.e. $\mathbf{x}^*$ is a common point for all operators $\mathbf{P}_i^\lambda$, where $i \in X_q$. If $X_q = J$, then the theorem is proved. Suppose that $X_q \neq J$. Since $\{I_k\}$ is a sequence of chaotic sets, for any k there is a minimal number r_k such that $q_k \le r_k$ and for any $q_k \le v \le r_k$ the sequence $\{I_v\}$ consists only of the set X_q, and the infinite sequence I_{r_k} contains if only one number $j \notin X_q$. Consider all subsets $Y_1, Y_2, \ldots, Y_t$ in J which contain the number j. Then there is if only one subset Y_s such that there is an infinite subsequence $\{s_k\}$ in $\{r_k\}$ such that $I_{s_k} = Y_s$. Then since $\mathbf{P}_i^\lambda\,\mathbf{x}^* = \mathbf{x}^*$ for $\forall\ i \in X_q$, one can build the following chain of inequalities:

$$\|\mathbf{x}^{(r_k)} - \mathbf{x}^*\| = \|\sum_{i \in X_q} \gamma_i^q \mathbf{P}_i^\lambda \mathbf{x}^{(r_k - 1)} - \mathbf{x}^*\| \le \|\mathbf{x}^{(r_k - 1)} - \mathbf{x}^*\| \le \ldots \le \|\mathbf{x}^{(q_k)} - \mathbf{x}^*\|$$

whence

$$\lim_{k \to \infty} \mathbf{x}^{(r_k)} = \mathbf{x}^*$$

Now taking into account that

$$\mathbf{x}^{(r_k + 1)} = \sum_{i \in Y_s} \gamma_i^s P_i^\lambda \mathbf{x}^{(r_k)}$$

there results

$$\alpha = \|\sum_{i \in Y_s} \gamma_i^s P_i^\lambda \mathbf{x}^* - \mathbf{z}\| = \|\mathbf{T}_s \mathbf{x}^* - \mathbf{z}\|$$

whence $\mathbf{P}_i^\lambda\,\mathbf{x}^* = \mathbf{x}^*$ for $\forall\ i \in Y_s$. Thus for a sequence $\{t_k\} = \{q_k\} \cup \{r_k\}$ we obtain that

$$\lim_{k \to \infty} \mathbf{x}^{(t_k)} = \mathbf{x}^*$$

and so $\mathbf{x}^*$ is a common point for all operators $\mathbf{P}_i^\lambda\,\mathbf{x}^* = \mathbf{x}^*$, where $i \in X_q \cup Y_s$. Continuing analogously one can show in a finite number of steps that $\mathbf{x}^*$ is a common point of operators $\mathbf{P}_i^\lambda$ for $\forall\ i \in J$, i.e. it is a solution of our system.

Replacing $\mathbf{z}$ by $\mathbf{x}^*$ in inequalities (12) we obtain that

$$\lim_{k \to \infty} \mathbf{x}^{(k)} = \mathbf{x}^*$$

So the sequence $\{\mathbf{x}^{(k)}\}$ also converges to the point $\mathbf{x}^*$.

Remark

In the case when a sequence of chaotic sets is an admissible sequence the corresponding reconstruction algorithm and the conditions of convergence of this algorithm for the case of arbitrary paracontracting operators are considered in the work [7].

Conclusions

The aim of this paper was to represent the algorithm AART which is an asynchronous implementation of the algebraic iterative algorithm ART and to give the sufficient conditions for convergence of this algorithm.

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