FRACTIONAL EULER-LAGRANGE EQUATIONS - NUMERICAL SOLUTIONS AND APPLICATIONS OF REFLECTION OPERATOR

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Abstract. In this work numerical solutions of fractional Euler-Lagrange equations describing free motion are considered. This type of equations contains a composition of left and right fractional derivatives. A reflection operator is applied to obtain relations between the Euler-Lagrange equations. In addition we verify the dependence between the respective numerical schemes using the same operator. In the final part of paper the examples of the numerical solutions are shown.

Introduction

The paper is devoted to the issue of numerical analysis of ordinary differential equations containing a composition of left and right fractional derivatives. This type of equations is obtained when the minimum action principle and fractional integration by parts rule are applied. There are many authors who considered a fractional Euler-Lagrange problem. Riewe in [1] investigated nonconservative Lagrangian and Hamiltonian mechanics and for those cases formulated a version of the Euler-Lagrange equations. On the other hand, Agrawal in [2-4] considered different types of variational problems, involving Riemann-Liouville, Caputo and Riesz fractional derivatives, respectively and he derived the corresponding Euler--Lagrange equations and discussed possibilities for describing boundary conditions in each case. Klimek in [5] proposed the sequential Lagrangian and Hamiltonian approaches to this problem. Other applications of fractional variational principles are presented in [6, 7]. The important problem is how to find the solutions of the fractional Euler-Lagrange equations. Using the fixed point theorems [8], one can obtain analytical results represented by a series of alternately left and right fractional integrals. Klimek in [9] showed an application of the Mellin transform for this problem, but this solution is represented by a series of special functions and therefore is difficult to use in practical calculations.

In references [10, 11] a numerical approach to solution of ordinary differential equations with left and right fractional derivatives is proposed. In our work, we shall present numerical solutions of the Euler-Lagrange equations and equations with the reflection operator.

1. Formulation of the problem

We consider the following fractional differential equations of order $\alpha \in (0, 1)$ (known in the literature as the fractional Euler-Lagrange equations [2-5, 8-11])

$$\left({}^{C}D_{b-}^{\alpha} D_{a+}^{\alpha} f\right)(t) = 0 \tag{1}$$

$$\left({}^{C}D_{a+}^{\alpha} D_{b-}^{\alpha} f\right)(t) = 0 \tag{2}$$

where operators D^{α} are the left and right fractional derivatives in Riemann-Liouville (3) and Caputo (4) senses defined as [12]

$$\left(D_{a+}^{\alpha} f\right)(t) = \left(D\, I_{a+}^{1-\alpha} f\right)(t), \qquad \left(D_{b-}^{\alpha} f\right)(t) = \left(-D\, I_{b-}^{1-\alpha} f\right)(t) \tag{3}$$

$$\left({}^{C}D_{a+}^{\alpha} f\right)(t) = \left(I_{a+}^{1-\alpha} D f\right)(t), \qquad \left({}^{C}D_{b-}^{\alpha} f\right)(t) = \left(-I_{b-}^{1-\alpha} D f\right)(t) \tag{4}$$

and operators I^{α} are fractional integrals of order α defined in [12]

$$\begin{aligned} \left(I_{a+}^{\alpha} f\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \quad \text{for } t > a \\ \left(I_{b-}^{\alpha} f\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} \mathrm{d}\tau, \quad \text{for } t < b \end{aligned} \tag{5}$$

The following relations between both definitions (3) and (4) take place [12]

$$\begin{aligned} \left(D_{a+}^{\alpha} f\right)(t) &= \left({}^{C}D_{a+}^{\alpha} f\right)(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a) \\ \left(D_{b-}^{\alpha} f\right)(t) &= \left({}^{C}D_{b-}^{\alpha} f\right)(t) + \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)} f(b) \end{aligned} \tag{6}$$

Equations (1) and (2) are supplemented by the adequate boundary conditions

$$f(a) = F_a, \quad f(b) = F_b \tag{7}$$

In this work we use the reflection operator Q on the interval $t \in [a,b]$

$$(Q\, f)(t) = f(a+b-t) \tag{8}$$

This operator acts on the fractional differential operators as follows [13]

$$
\begin{aligned}
\left(Q\, {}^{C}D^{\alpha}_{a+} f\right)(t) &= \left({}^{C}D^{\alpha}_{b-} Q f\right)(t) \\
\left(Q\, {}^{C}D^{\alpha}_{b-} f\right)(t) &= \left({}^{C}D^{\alpha}_{a+} Q f\right)(t) \\
\left(Q D^{\alpha}_{a+} f\right)(t) &= \left(D^{\alpha}_{b-} Q f\right)(t) \\
\left(Q D^{\alpha}_{b-} f\right)(t) &= \left(D^{\alpha}_{a+} Q f\right)(t)
\end{aligned}
\tag{9}
$$

2. Numerical solution

In order to develop a discrete form of equations (1) and (2), the homogenous grid of nodes is introduced

$$
a = t_0 < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < t_N = b, \qquad t_i = t_0 + i\,\Delta t \tag{10}
$$

A value of function f at the moment of time t_i is denoted as $f_i = f(t_i)$.

2.1. Discrete form of equation (1)

At first we determine numerical schemes for both fractional operators occurring in eq. (1). The value of derivative (6) (internal operator) at the moment of time t_i can be approximated as [14]

$$
\begin{aligned}
\left(D^{\alpha}_{a+} f\right)(t)\Big|_{t=t_i} &= f_0 \frac{(t_i - t_0)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_i} \frac{f'(\tau)}{(t_i-\tau)^{\alpha}}\, d\tau \\
&\cong f_0 \frac{(t_i - t_0)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{f_{j+1} - f_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(t_i-\tau)^{\alpha}}\, d\tau \\
&= f_0 \frac{(t_i - t_0)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{f_{j+1} - f_j}{\Delta t} \cdot \frac{(t_i - t_j)^{1-\alpha} - (t_i - t_{j+1})^{1-\alpha}}{1-\alpha} \\
&= (\Delta t)^{-\alpha} \sum_{j=0}^{i} f_j\, v_1(i,j)
\end{aligned}
\tag{11}
$$

where

$$
v_1(i,j) = \frac{1}{\Gamma(2-\alpha)}
\begin{cases}
(1-\alpha) i^{-\alpha} + (i-1)^{1-\alpha} - i^{1-\alpha} & \text{for } j = 0 \\
(i-j+1)^{1-\alpha} - 2(i-j)^{1-\alpha} + (i-j-1)^{1-\alpha} & \text{for } j = 1,\ldots,i-1 \\
1 & \text{for } j = i
\end{cases}
\tag{12}
$$

Substituting $g(t)=\left(D_{a+}^{\alpha} f\right)(t)$ in eq. (1), we can directly discretise the composition of operators

$$\begin{aligned}
\left({}^{C}D_{b-}^{\alpha} D_{a+}^{\alpha} f\right)(t)\Big|_{t=t_i} &= \left({}^{C}D_{b-}^{\alpha} g\right)(t)\Big|_{t=t_i} = \frac{-1}{\Gamma(1-\alpha)} \int_{t_i}^{t_N} \frac{g'(\tau)}{(\tau-t_i)^{\alpha}} d\tau \\
&\cong \frac{-1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{g_{j+1}-g_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(\tau-t_i)^{\alpha}} d\tau \\
&= \frac{-1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{g_{j+1}-g_j}{\Delta t} \cdot \frac{\left(t_{j+1}-t_i\right)^{1-\alpha}-\left(t_j-t_i\right)^{1-\alpha}}{1-\alpha} \\
&= (\Delta t)^{-\alpha} \sum_{j=i}^{N} g_j w_1(i,j)
\end{aligned} \tag{13}$$

where

$$w_1(i,j)=\frac{1}{\Gamma(2-\alpha)}\begin{cases} 1 & \text{for } j=i \\ (j-i+1)^{1-\alpha}-2(j-i)^{1-\alpha}+(j-i-1)^{1-\alpha} & \text{for } j=i+1,\ldots,N-1 \\ (N-i-1)^{1-\alpha}-(N-i)^{1-\alpha} & \text{for } j=N \end{cases} \tag{14}$$

Next, substituting the discrete form of derivative (11) into (13), the following form is obtained

$$\left({}^{C}D_{b-}^{\alpha} D_{a+}^{\alpha} f\right)(t)\Big|_{t=t_i} \cong (\Delta t)^{-2\alpha} \sum_{j=i}^{N} \left[w_1(i,j) \sum_{k=0}^{j} v_1(j,k) f_k \right] \tag{15}$$

In order to solve eq. (1) numerically with boundary conditions (7), it is necessary to solve the following system of algebraic equations

$$\begin{aligned}
& f_0 = F_a \\
& (\Delta t)^{-2\alpha} \sum_{j=i}^{N} \left[w_1(i,j) \sum_{k=0}^{j} v_1(j,k) f_k \right] = 0 \quad \text{for } i=1,\ldots,N-1 \\
& f_N = F_b
\end{aligned} \tag{16}$$

2.2. Discrete form of equation (2)

This method is similar to the previous case. At the beginning we discretise the operator $\left(D_{b-}^{\alpha} f\right)(t)$ at time t_i as

$$
\begin{aligned}
D_{b-}^{\alpha} f(t)\Big|_{t=t_i} &= f_N \frac{(t_N - t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{-1}{\Gamma(1-\alpha)} \int_{t_i}^{t_N} \frac{f'(\tau)}{(\tau - t_i)^{\alpha}} d\tau \\
&\cong f_N \frac{(t_N - t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{-1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{f_{j+1} - f_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(\tau - t_i)^{\alpha}} d\tau \\
&= f_N \frac{(t_N - t_i)^{-\alpha}}{\Gamma(1-\alpha)} + \frac{-1}{\Gamma(1-\alpha)} \sum_{j=i}^{N-1} \frac{f_{j+1} - f_j}{\Delta t} \cdot \frac{(t_{j+1} - t_i)^{1-\alpha} - (t_j - t_i)^{1-\alpha}}{1-\alpha} \\
&= (\Delta t)^{-\alpha} \sum_{j=i}^{N} f_j v_2(i, j)
\end{aligned}
\tag{17}
$$

where

$$
v_2(i, j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} 1 & \text{for } j = i \\ (j - i + 1)^{1-\alpha} - 2(j - i)^{1-\alpha} + (j - i - 1)^{1-\alpha} & \text{for } j = i + 1, \ldots, N - 1 \\ (1-\alpha)(N - i)^{-\alpha} + (N - i - 1)^{1-\alpha} - (N - i)^{1-\alpha} & \text{for } j = N \end{cases}
\tag{18}
$$

and next

$$
\begin{aligned}
\left({}^{C}D_{a+}^{\alpha} D_{b-}^{\alpha} f\right)(t)\Big|_{t=t_i} = \left({}^{C}D_{a+}^{\alpha} g\right)(t)\Big|_{t=t_i} &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_i} \frac{g'(\tau)}{(t_i - \tau)^{\alpha}} d\tau \\
&\cong \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{g_{j+1} - g_j}{\Delta t} \int_{t_j}^{t_{j+1}} \frac{1}{(t_i - \tau)^{\alpha}} d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{i-1} \frac{g_{j+1} - g_j}{\Delta t} \cdot \frac{(t_i - t_j)^{1-\alpha} - (t_i - t_{j+1})^{1-\alpha}}{1-\alpha} \\
&= (\Delta t)^{-\alpha} \sum_{j=0}^{i} g_j w_2(i, j)
\end{aligned}
\tag{19}
$$

where

$$
w_2(i, j) = \frac{1}{\Gamma(2-\alpha)} \begin{cases} (i-1)^{1-\alpha} - i^{1-\alpha} & \text{for } j = 0 \\ (i - j + 1)^{1-\alpha} - 2(i - j)^{1-\alpha} + (i - j - 1)^{1-\alpha} & \text{for } j = 1, \ldots, i - 1 \\ 1 & \text{for } j = i \end{cases}
\tag{20}
$$

The discrete form for the operators composition in eq. (2) we can write (after substitution the discrete form of (17) into (19)) as follows

$$\left({}^{C}D^{\alpha}_{a+}\,D^{\alpha}_{b-}\,f\right)(t)\Big|_{t=t_i} \cong (\Delta t)^{-2\alpha}\sum_{j=0}^{i}\left[w_2(i,j)\sum_{k=j}^{N}v_2(j,k)\,f_k\right] \tag{21}$$

Similar to the previous equation, it is necessary to solve the following system of equations

$$\begin{aligned} &f_0 = F_a \\ &(\Delta t)^{-2\alpha}\sum_{j=0}^{i}\left[w_2(i,j)\sum_{k=j}^{N}v_2(j,k)\,f_k\right] = 0 \quad \text{for } i = 1,\dots,N-1 \\ &f_N = F_b \end{aligned} \tag{22}$$

2.3. Numerical solutions of equations with the reflection operator

In this subsection we present an application of reflection operator Q (8) acting on eq. (1)

$$\left(Q\,{}^{C}D^{\alpha}_{b-}\,D^{\alpha}_{a+}\,f\right)(t) = 0 \tag{23}$$

Taking into account relations (9), one can write eq. (1) in the following form

$$\left({}^{C}D^{\alpha}_{a+}\,D^{\alpha}_{b-}\,Q\,f\right)(t) = 0 \tag{24}$$

Now, we show identity of equations (23) and (24) using numerical approach. For this purpose we present numerical solutions which are determined using schemes (16) and (22). Let us note that operator Q acts as follows

$$\begin{aligned} &Q\,t\big|_{t=t_i} = Q\,t_i = t_{N-i} \\ &(Qf)(t)\big|_{t=t_i} = Q\,f_i = f_{N-i} \end{aligned} \tag{25}$$

Then, we get

$$\begin{aligned} \left(Q\,{}^{C}D^{\alpha}_{b-}\,D^{\alpha}_{a+}\,f\right)(t)\Big|_{t=t_i} &\cong Q\left((\Delta t)^{-2\alpha}\sum_{j=i}^{N}\left[w_1(i,j)\sum_{k=0}^{j}v_1(j,k)\,f_k\right]\right) \\ &= (\Delta t)^{-2\alpha}\sum_{j=N-i}^{N}\left[w_1(N-i,j)\sum_{k=0}^{j}v_1(j,k)\,f_k\right] \end{aligned} \tag{26}$$

After changing the order of summation we can write the above formula as

$$\left(Q\,{}^{C}D^{\alpha}_{b-}\,D^{\alpha}_{a+}\,f\right)(t)\Big|_{t=t_i} \cong (\Delta t)^{-2\alpha}\sum_{j=0}^{i}\left[w_1(N-i,N-j)\sum_{k=j}^{N}v_1(N-j,N-k)\,f_{N-k}\right] \tag{27}$$

Hence, we change in scheme (21) only term f_k by f_{N-k} and we obtain

$$\left({}^{C}D^{\alpha}_{a+}\, D^{\alpha}_{b-}\, Q\, f\right)(t)\Big|_{t=t_i} \cong (\Delta t)^{-2\alpha} \sum_{j=0}^{i}\left[w_2(i,j) \sum_{k=j}^{N} v_2(j,k)\, f_{N-k} \right] \tag{28}$$

Also, in both equations (23) and (24) the values of boundary conditions must be replaced: $f(a) = F_b$ and $f(b) = F_a$ (in numerical approach: $f_0 = F_b$ and $f_N = F_a$).

Analysing the values of weight at f_{N-k} in (27) and (28), one can note that they are equal and both schemes are equivalent. One can affirm that the following relations between coefficients occurr

$$w_2(i,j) = w_1(N-i, N-j), \qquad v_2(i,j) = v_1(N-i, N-j) \tag{29}$$

In a similar way one can obtain other schemes for equations in which other compositions of fractional differential operators appear.

3. Examples of computations

In this section the numerical results of calculations are presented. In presented solutions of equations the following parameters have been assumed: $a = 0$, $b = 1$, $\alpha = \{0.1, 0.3, 0.5, 0.7, 0.9, 0.999\}$, $N = 1000$. The values of boundary conditions are following: $F_a = 0$, $F_b = 1$ for eq. (1) and $F_a = 1$, $F_b = 0$ for eq. (2).

In Figure 1 the solutions of equations (1) and (2) for different values of the parameter α are presented. One can see that both solutions are symmetrical.

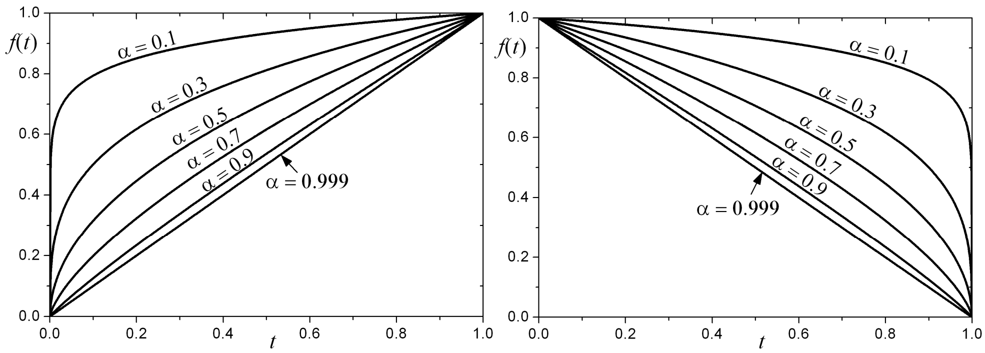

Fig. 1. Solutions of eq. (1) (left-side) and eq. (2) (right-side)

Conclusions

In this work the fractional Euler-Lagrange equations were considered. This type of equation includes a composition of the left and the right fractional derivatives. The analytical solutions of these equations are difficult to apply in practical calculations. Numerical solution is an alternative approach to the analytical one. In this study the numerical schemes were presented to obtain the solution for two cases of the fractional Euler-Lagrange equations. The considered equations are related via the reflection operator. This relationship was also proved for numerical schemes. Analysing solutions presented in Figure 1 we observe that the solutions of equation (1) are a symmetrical reflection of the solutions of equation (2).

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