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ON REFLECTION SYMMETRY IN FRACTIONAL MECHANICS

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Abstract. We study the properties of fractional differentiation with respect to reflection mapping in a finite interval. The symmetric and anti-symmetric fractional derivatives in a full interval are expressed as fractional differential operators in left or right subintervals obtained by subsequent partitions. These representation properties and the reflection symmetry of the action and variation are applied to derive Euler-Lagrange equations of fractional free motion. Then the localization phenomenon for these equations is discussed.

Introduction

Fractional derivatives appear in differential equations modeling many processes in physics, mechanics, control theory, biochemistry, bioengineering and economics. The theory of fractional differential equations (FDEs) is an area of investigations that has developed rapidly during recent decades and established a meaningful field of pure and applied mathematics. Monographs [1-6] enclose a review of solving methods using analytical and numerical approaches. Recently, also equations including both left and right fractional derivatives have been discussed in papers [7-12]. Such differential equations mixing both types of derivatives naturally emerge in fractional mechanics whenever standard variational calculus is applied in the derivation of Euler-Lagrange equations.

This approach was started in 1996 by Riewe [13, 14], developed by Agrawal and Klimek [15-17] and has been investigated ever since (compare papers [18-30] and the references therein).

The known existence results for equations with left- and right-sided derivatives lead to a solution with some additional restrictions including the parameters of the problem such as the order of derivatives and the length of the time interval [7--10]. For instance, a detailed discussion of the existence conditions for the solution of the fractional oscillator equation has been given in [11]. Here, we shall show that in fact such equations can be localized in subintervals - left or right, at least in the case of free motion.

The paper is organized as follows. In the next section we recall the basic definitions and properties of both integrals and derivatives in a finite interval. Then, for a function determined in such an interval, we define the symmetric and anti-

-symmetric derivatives of the Riemann-Liouville and Caputo type. The main results are given in Section 2 where we study the reflection symmetry properties of fractional differentiation. The obtained results are applied to a model of fractional free motion in Section 3. We derive Euler-Lagrange equations for the reflection symmetric and anti-symmetric parts of the trajectory using the methods introduced in [32]. It appears that the obtained system of equations for the symmetric and anti-symmetric parts of the trajectory in interval [0,*b*], it is in fact [0,*b/*2] - or respectively [*b/*2,*b*] - localized. The paper is closed by a short discussion of possible extension of the obtained results and their further application.

1. Reflection symmetry in fractional calculus

First we recall the basic definitions of fractional calculus [1, 31].

Definition 1.1. Let $Re(\alpha) > 0$. The left- and respectively right-sided Riemann-Liouville integrals of order α are given by formulas

$$(I^{\alpha}_{0+}f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t>0 \tag{1}$$

$$(I^{\alpha}_{b-}f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1} f(s)ds, \quad t<b, \tag{2}$$

where Γ denotes the Euler gamma function.

Definition 1.2. Let $Re(\alpha) \in (n-1,n)$. The left- and right-sided Riemann-Liouville derivatives of order α are defined as

$$(D^{\alpha}_{0+}f)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{0+}f)(t), \;\; t>0 \tag{3}$$

$$(D^{\alpha}_{b-}f)(t) = \left(-\frac{d}{dt}\right)^n (I^{n-\alpha}_{b-}f)(t), \;\; t<b \tag{4}$$

Analogous formulas yield the left- and right-sided Caputo derivatives of order α

$$({}^{c}D^{\alpha}_{0+}f)(t) = I^{n-\alpha}_{0+}\left(\frac{d}{dt}\right)^n f(t), \;\; t>0 \tag{5}$$

$$({}^{c}D^{\alpha}_{b-}f)(t) = I^{n-\alpha}_{b-}\left(-\frac{d}{dt}\right)^n f(t), \;\; t<b \tag{6}$$

Definition 1.3. Let $Re(\alpha) \in (n-1,n)$. The symmetric and respectively anti-symmetric Riemann-Liouville derivatives in interval $[0,b]$ are given as follows:

$$\mathcal{D}^{\alpha}_{[0,b]} := \frac{1}{2}[D^{\alpha}_{0+} + (-1)^{n} D^{\alpha}_{b-}] \tag{7}$$

$$\bar{\mathcal{D}}^{\alpha}_{[0,b]} := \frac{1}{2}[D^{\alpha}_{0+} + (-1)^{n-1} D^{\alpha}_{b-}]. \tag{8}$$

The symmetric and respectively anti-symmetric Caputo derivatives in interval $[0, b]$ are given as:

$${}^{c}\mathcal{D}^{\alpha}_{[0,b]} := \frac{1}{2}[\,{}^{c}D^{\alpha}_{0+} + (-1)^{n}\,{}^{c}D^{\alpha}_{b-}] \tag{9}$$

$${}^{c}\bar{\mathcal{D}}^{\alpha}_{[0,b]} := \frac{1}{2}[\,{}^{c}D^{\alpha}_{0+} + (-1)^{n-1}\,{}^{c}D^{\alpha}_{b-}] \tag{10}$$

Let us also introduce the notion of fractional integrals over finite interval $[a, b]$:

$$I^{n-\alpha}_{[a,b]} f(t) := \frac{1}{2\Gamma(n-\alpha)} \int_{a}^{b} |t-s|^{n-1-\alpha} f(s)ds \tag{11}$$

$$\bar{I}^{n-\alpha}_{[a,b]} f(t) := \frac{1}{2\Gamma(n-\alpha)} \int_{a}^{b} |t-s|^{n-1-\alpha}\, \mathrm{sgn}(t-s) f(s)ds. \tag{12}$$

It is easy to check that the symmetric and anti-symmetric fractional derivatives can be represented similar to the one-sided Riemann-Liouville and Caputo operators.

Proposition 1.4. Let $Re(\alpha) \in (n-1, n)$. The symmetric and anti-symmetric Riemann-Liouville derivatives in interval $[a,b]$ obey relations

$$\mathcal{D}^{\alpha}_{[a,b]} := \left(\frac{d}{dt}\right)^{n} I^{n-\alpha}_{[a,b]} \tag{13}$$

$$\bar{\mathcal{D}}^{\alpha}_{[a,b]} := \left(\frac{d}{dt}\right)^{n} \bar{I}^{n-\alpha}_{[a,b]}. \tag{14}$$

The symmetric and anti-symmetric Caputo derivatives in interval $[a,b]$ obey relations

$${}^{c}\mathcal{D}^{\alpha}_{[a,b]} := I^{n-\alpha}_{[a,b]} \left(\frac{d}{dt}\right)^{n} \tag{15}$$

$${}^{c}\bar{\mathcal{D}}^{\alpha}_{[a,b]} := \bar{I}^{n-\alpha}_{[a,b]} \left(\frac{d}{dt}\right)^{n}. \tag{16}$$

Definition 1.5. Reflection operators $Q_{[0,b]}, Q_{[0,\frac{b}{2^m}]}$ and $Q_{[b-\frac{b}{2^m},\ b]}$, acting on arbitrary function f determined in interval $[0, b]$ are given as follows:

$$Q_{[0,b]} f(t) := f(b-t) \tag{17}$$

$$Q_{[0,\frac{b}{2^m}]}f(t) := f(\frac{b}{2^m} - t) \tag{18}$$

$$Q_{[b-\frac{b}{2^m},b]}f(t) := f(\frac{2^{m+1}-1}{2^m}b - t). \tag{19}$$

Definition 1.6. Let f be an arbitrary function determined in $[0,b]$ and vector $[J] = [j_1, \dots, j_m]$ have components in the two-element set $\{0,1\}$. The following recursive formulas define the respective projections/components of function f:

$$f_{[j]}(t) := \frac{1}{2}\left(1 + (-1)^j Q_{[0,b]}\right)f(t) \tag{20}$$

$$f_{[J,j_{m+1}]}(t) := \begin{cases} \frac{1}{2}\left(1 + (-1)^{j_{m+1}} Q_{[0,\frac{b}{2^m}]}\right)f_{[J]}(t) & t \le \frac{b}{2^m} \\ \frac{1}{2} f_{[J]}(t) & t > \frac{b}{2^m} \end{cases} \tag{21}$$

$$f_{[j_{m+1},J]}(t) := \begin{cases} \frac{1}{2} f_{[J]}(t) & t < b - \frac{b}{2^m} \\ \frac{1}{2}\left(1 + (-1)^{j_{m+1}} Q_{[b-\frac{b}{2^m},b]}\right)f_{[J]}(t) & t \ge b - \frac{b}{2^m} \end{cases} \tag{22}$$

For any $m \in N$ function f can be split into the respective projections

$$f_{[J]}(t) = \sum_{j_{m+1}=0}^{1} f_{[j_{m+1},J]}(t) \tag{23}$$

$$f_{[J]}(t) = \sum_{j_{m+1}=0}^{1} f_{[J,j_{m+1}]}(t) \tag{24}$$

$$f(t) = \sum_{[J]} f_{[J]}(t), \tag{25}$$

where the summation in (25) is over all the *m* - component vectors with coordinates in the two-element set {0, 1}.

Property 1.7. *L*et $f_{[j]}$ be the projection given in Definition 1.6.

(1) The following relations are valid:

$$Q_{[0,b]}f_{[j]}(t) = f_{[j]}(b - t) = (-1)^j f_{[j]}(t). \tag{26}$$

(2) The projections fulfill orthogonality relations

$$\int_0^b f_{[j]}(t)\, g_{[k]}(t)dt = \left(\int_0^b f_{[j]}(t)\, g_{[j]}(t)\, dt\right)\delta_{j,k}. \tag{27}$$

2. Properties of the symmetric and anti-symmetric derivatives

In this section we shall discuss the representation properties of the symmetric and anti-symmetric fractional derivatives of order $\alpha \in (1,2)$. We shall prove that acting on the $[J]$ - projections of function f they can be expressed as operators dependent on the values of function in a relatively short interval obtained as a result of the corresponding partitions of $[0, b]$.

The representation properties enclosed in Propositions 1.1 and 1.3 were proved in [32]. They show the connection between the fractional derivatives determined in interval [0,*b*] and the ones determined over subintervals $[0, b/2]$ and $[0, b/2^m]$ respectively. The new results given in Propositions 1.2 and 1.4, connect the derivatives determined in [0,*b*] with those determined over the right subintervals.

Proposition 2.1. Let $f_{[j]}$ be the $[j]$-projection of function f given by formula (20). Its symmetric derivatives of order α $\in (1,2)$ in interval $[0, b]$ can be represented as follows:

$$\mathcal{D}^{\alpha}_{[0,b]} f_{[j]}(t) = (1 + (-1)^j Q_{[0,b]}) \mathcal{D}^{\alpha}_{[0,b/2]} f_{[j]}(t) \tag{28}$$

$${}^{c}\mathcal{D}^{\alpha}_{[0,b]} f_{[j]}(t) = (1 + (-1)^j Q_{[0,b]})\, {}^{c}\mathcal{D}^{\alpha}_{[0,b/2]} f_{[j]}(t). \tag{29}$$

Let $f_{[J]}$ be the $[J]$-projection of function f given by formula (21) for vector $[J] = [j_1, \dots, j_m]$, $j_l \in \{0,1\}$. Its symmetric derivatives of order α $\in (1,2)$ in interval $[0, b]$ can be represented as follows:

$$\mathcal{D}^{\alpha}_{[0,b]} f_{[J]}(t) = 2^m \Pi_{[J]} \mathcal{D}^{\alpha}_{[0,b/2^m]} f_{[J]}(t) \tag{30}$$

$${}^{c}\mathcal{D}^{\alpha}_{[0,b]} f_{[J]}(t) = 2^m \Pi_{[J]}\, {}^{c}\mathcal{D}^{\alpha}_{[0,b/2^m]} f_{[J]}(t), \tag{31}$$

where we denoted as $\Pi_{[J]}$ the ordered product of the projection operators

$$\Pi_{[J]} := 2^{-m}(1 + (-1)^{j_1} Q_{[0,b]}) \dots \left(1 + (-1)^{j_m} Q_{\left[0,\frac{b}{2^{m-1}}\right]}\right) \tag{32}$$

Proposition 2.2. Let $f_{[j]}$ be the $[j]$ - projection of function f given by formula (20). Its symmetric derivatives of order α $\in (1,2)$ in interval $[0, b]$ can be represented as follows:

$$\mathcal{D}^{\alpha}_{[0,b]} f_{[j]}(t) = (1 + (-1)^j Q_{[0,b]}) \mathcal{D}^{\alpha}_{[b/2,b]} f_{[j]}(t) \tag{33}$$

$${}^{c}\mathcal{D}^{\alpha}_{[0,b]} f_{[j]}(t) = (1 + (-1)^j Q_{[0,b]})\, {}^{c}\mathcal{D}^{\alpha}_{[b/2,b]} f_{[j]}(t). \tag{34}$$

Let $f_{[J]}$ be the $[J]$-projection of function f given by formula (22) for vector $[J] = [j_1, \dots, j_m]$, $j_l \in \{0,1\}$. Its symmetric derivatives of order α ∈ $(1,2)$ in interval $[0, b]$ can be represented as follows:

$$\mathcal{D}^{\alpha}_{[0,b]} f_{[J]}(t) = 2^m \prod^1_{[J]} \mathcal{D}^{\alpha}_{[b-\frac{b}{2^m},b]} f_{[J]}(t) \tag{35}$$

$$^c\mathcal{D}^{\alpha}_{[0,b]} f_{[J]}(t) = 2^m \prod^1_{[J]} {}^c\mathcal{D}^{\alpha}_{[b-\frac{b}{2^m},b]} f_{[J]}(t), \tag{36}$$

where we denoted as $\prod^1_{[J]}$ the following ordered product of the projection operators

$$\prod^1_{[J]} := 2^{-m}\left(1 + (-1)^{j_1} Q_{[0,b]}\right) \dots \left(1 + (-1)^{j_m} Q_{\left[b-\frac{b}{2^{m-1}},b\right]}\right). \tag{37}$$

Proof. First, we check property (33) using the integration properties and the reflection properties of the second order derivative:

$$\mathcal{D}^{\alpha}_{[0,b]} f_{[j]}(t) = \frac{1}{2\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^b \frac{f_{[j]}(s)ds}{|t-s|^{\alpha-1}} =$$

$$= \frac{1}{2\Gamma(2-\alpha)} \frac{d^2}{dt^2} \left[\int_0^{\frac{b}{2}} \frac{f_{[j]}(s)ds}{|t-s|^{\alpha-1}} + \int_{\frac{b}{2}}^b \frac{f_{[j]}(s)ds}{|t-s|^{\alpha-1}} \right] =$$

$$= \begin{bmatrix} b - s = w, & ds = -dw \\ f_{[j]}(b-w) = (-1)^j f_{[j]}(w) \end{bmatrix}$$

$$= \frac{1}{2\Gamma(2-\alpha)} \frac{d^2}{dt^2} \left[(-1)^j \left[\int_{\frac{b}{2}}^b \frac{f_{[j]}(w)dw}{|b-t-w|^{\alpha-1}} + \int_{\frac{b}{2}}^b \frac{f_{[j]}(s)ds}{|t-s|^{\alpha-1}} \right] \right] =$$

$$= \left((-1)^j Q_{[0,b]} + 1\right) \frac{1}{2\Gamma(2-\alpha)} \frac{d^2}{dt^2} I^{2-\alpha}_{[b/2,b]} f_{[j]}(t) =$$

$$= \left((-1)^j Q_{[0,b]} + 1\right) \mathcal{D}^{\alpha}_{[b/2,b]} f_{[j]}(t).$$

Let us observe that equation (33) remains valid when we replace 0 by $b - \frac{b}{2^m}$ and take $t \in [b - \frac{b}{2^m}, b]$:

$$\mathcal{D}^{\alpha}_{\left[b-\frac{b}{2^m},b\right]} f_{[j_{m+1},J]}(t)|_{t\in[b-\frac{b}{2^m},b]} =$$

$$= \left((-1)^{j_{m+1}} Q_{\left[b-\frac{b}{2^m}, b\right]} + 1\right) \mathcal{D}^{\alpha}_{\left[b-\frac{b}{2^{m+1}}, b\right]} f_{[j_{m+1}, J]}(t).$$

Thus, we can prove property (35) by means of the mathematical induction principle. Using (33) and (35) we obtain for the $[j_{m+1}, J]$-projection

$$\mathcal{D}^{\alpha}_{[0,b]} f_{[j_{m+1}, J]}(t) = 2^m \prod\nolimits_{[J]}^{1} \mathcal{D}^{\alpha}_{\left[b-\frac{b}{2^m}, b\right]} f_{[j_{m+1}, J]}(t) =$$

$$= 2^m \prod\nolimits_{[J]}^{1} \left((-1)^{j_{m+1}} Q_{\left[b-\frac{b}{2^m}, b\right]} + 1\right) \mathcal{D}^{\alpha}_{\left[b-\frac{b}{2^{m+1}}, b\right]} f_{[j_{m+1}, J]}(t) =$$

$$= 2^{m+1} \prod\nolimits_{[j_{m+1}, J]}^{1} \mathcal{D}^{\alpha}_{\left[b-\frac{b}{2^{m+1}}, b\right]} f_{[j_{m+1}, J]}(t)$$

which proves formula (35) to be valid for arbitrary $m \in N$. The calculations for properties (34) and (36) are similar.

Proposition 2.3. Let $f_{[j]}$ be the $[j]$ - projection of function f given by formula (20). Its anti-symmetric derivatives of order $\alpha \in (1,2)$ in interval $[0, b]$ can be represented as follows:

$$\bar{\mathcal{D}}^{\alpha}_{[0,b]} f_{[j]}(t) = (1 + (-1)^{j+1} Q_{[0,b]}) \bar{\mathcal{D}}^{\alpha}_{[0,b/2]} f_{[j]}(t) \tag{38}$$

$$^{c}\bar{\mathcal{D}}^{\alpha}_{[0,b]} f_{[j]}(t) = (1 + (-1)^{j+1} Q_{[0,b]})\ {}^{c}\bar{\mathcal{D}}^{\alpha}_{[0,b/2]} f_{[j]}(t). \tag{39}$$

Let $f_{[J]}$ be the $[J]$-projection of function f given by formula (21) for vector $[J] = [j_1, \dots, j_m]$, $j_i \in \{0,1\}$. Its anti-symmetric derivatives of order $\alpha \in (1,2)$ in interval $[0, b]$ can be represented as follows:

$$\bar{\mathcal{D}}^{\alpha}_{[0,b]} f_{[J]}(t) = 2^m \prod\nolimits^{-}_{[J]} \bar{\mathcal{D}}^{\alpha}_{[0,b/2^m]} f_{[J]}(t) \tag{40}$$

$$^{c}\bar{\mathcal{D}}^{\alpha}_{[0,b]} f_{[J]}(t) = 2^m \prod\nolimits^{-}_{[J]}\ {}^{c}\bar{\mathcal{D}}^{\alpha}_{\left[0,\frac{b}{2^m}\right]} f_{[J]}(t). \tag{41}$$

where we denoted $\prod^{-}_{[J]}$ as the ordered product of the projection operators

$$\prod\nolimits^{-}_{[J]} := 2^{-m}(1 + (-1)^{j_1+1} Q_{[0,b]}) \dots \left(1 + (-1)^{j_m+1} Q_{\left[0,\frac{b}{2^{m-1}}\right]}\right). \tag{42}$$

Proposition 2.4. Let $f_{[j]}$ be the $[j]$ - projection of function f given by formula (20). Its anti-symmetric derivatives of order $\alpha \in (1,2)$ in interval $[0, b]$ can be represented as follows:

$$\bar{\mathcal{D}}^{\alpha}_{[0,b]} f_{[j]}(t) = (1 + (-1)^{j+1} Q_{[0,b]}) \bar{\mathcal{D}}^{\alpha}_{[b/2,b]} f_{[j]}(t) \tag{43}$$

$$^{c}\bar{\mathcal{D}}^{\alpha}_{[0,b]} f_{[j]}(t) = (1 + (-1)^{j+1} Q_{[0,b]})\ ^{c}\bar{\mathcal{D}}^{\alpha}_{[b/2,b]} f_{[j]}(t). \tag{44}$$

Let $f_{[J]}$ be the $[J]$-projection of function f given by formula (22) for vector $[J] = [j_1, \dots, j_m]$, $j_i \in \{0,1\}$. Its anti-symmetric derivatives of order $\alpha \in (1,2)$ in interval $[0, b]$ can be represented as follows

$$\bar{\mathcal{D}}^{\alpha}_{[0,b]} f_{[J]}(t) = 2^m \Pi^{-1}_{[J]} \bar{\mathcal{D}}^{\alpha}_{[b-\frac{b}{2^m},b]} f_{[J]}(t) \tag{45}$$

$$^{c}\bar{\mathcal{D}}^{\alpha}_{[0,b]} f_{[J]}(t) = 2^m \Pi^{-1}_{[J]}\ ^{c}\bar{\mathcal{D}}^{\alpha}_{[b-\frac{b}{2^m},b]} f_{[J]}(t), \tag{46}$$

where we denoted as $\Pi^{-1}_{[J]}$ the ordered product of the projection operators

$$\Pi^{-1}_{[J]} := 2^{-m}\left(1 + (-1)^{j_1+1} Q_{[0,b]}\right) \dots \left(1 + (-1)^{j_m+1} Q_{[b-\frac{b}{2^{m-1}},b]}\right). \tag{47}$$

Now, we apply the representation properties of symmetric and anti-symmetric fractional derivatives of order $\alpha \in (1,2)$ to prove some integration formulas. We quote the proposition below from paper [32].

Proposition 2.5. Let $\alpha \in (1,2)$. The following integration formulas are valid for any pair of functions so that $f \in L_1(0, b)$ and $\mathcal{D}^{\alpha}_{[0,b]} g \in L_1(0, b)$ or $^{c}\mathcal{D}^{\alpha}_{[0,b]} g \in L_1(0, b)$ respectively

$$\int_0^b f(t) \mathcal{D}^{\alpha}_{[0,b]} g(t) dt = \sum_{j=0}^{1} \int_0^b f_{[j]}(t)\ \mathcal{D}^{\alpha}_{[0,b]} g_{[j]}(t) dt = \tag{48}$$

$$= 2 \sum_{j=0}^{1} \int_0^b f(t)_{[j]} \mathcal{D}^{\alpha}_{[0,b/2]} g_{[j]}(t) dt$$

$$\int_0^b f(t)\ ^{c}\mathcal{D}^{\alpha}_{[0,b]} g(t) dt = \sum_{j=0}^{1} \int_0^b f_{[j]}(t)\ ^{c}\mathcal{D}^{\alpha}_{[0,b]} g_{[j]}(t) dt = \tag{49}$$

$$= 2 \sum_{j=0}^{1} \int_0^b f(t)_{[j]}\ ^{c}\mathcal{D}^{\alpha}_{[0,b/2]} g_{[j]}(t) dt.$$

The next result is an analogue of Proposition 2.5.

Proposition 2.6. Let $\alpha \subset (1,2)$. The following integration formulas are valid for any pair of functions so that $f \in L_1(0, b)$ and $\mathcal{D}^{\alpha}_{[0,b]} g \in L_1(0, b)$ or $^{c}\mathcal{D}^{\alpha}_{[0,b]} g \in L_1(0, b)$ respectively

$$\int_0^b f(t) \mathcal{D}^{\alpha}_{[0,b]} g(t) dt = \sum_{j=0}^{1} \int_0^b f_{[j]}(t)\ \mathcal{D}^{\alpha}_{[0,b]} g_{[j]}(t) dt = \tag{50}$$

$$= 2\sum_{j=0}^{1}\int_0^b f\,(t)_{[j]}\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)\,dt.$$

$$\int_0^b f(t)\ {}^{c}\mathcal{D}^{\alpha}_{[0,b]}g(t)dt = \sum_{j=0}^{1}\int_0^b f_{[j]}(t)\ \ {}^{c}\mathcal{D}^{\alpha}_{[0,b]}g_{[j]}(t)dt = \tag{51}$$

$$= 2\sum_{j=0}^{1}\int_0^b \quad f_{[j]}(t)\ {}^{c}\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)\,dt.$$

Proof. Formula (50) results from the reflection symmetry of the integral in $[0,b]$ and the representation property of the symmetric Riemann-Liouville fractional derivative given in (7)

$$\int_0^b f(t)\mathcal{D}^{\alpha}_{[0,b]}g(t)dt = \ \sum_{j=0}^{1}\int_0^b f_{[j]}(t)\ \ \mathcal{D}^{\alpha}_{[0,b]}g_{[j]}(t)dt =$$

$$= \sum_{j=0}^{1}\int_0^b f_{[j]}\left(1+(-1)^{j}Q_{[0,b]}\right)\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)dt =$$

$$= \ \sum_{j=0}^{1}\int_0^b f_{[j]}\,\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)dt +$$

$$+(-1)^{j}\sum_{j=0}^{1}\int_0^b f_{[j]}\,Q_{[0,b]}\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)dt =$$

$$= \sum_{j=0}^{1}\int_0^b f_{[j]}\,\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)dt +$$

$$+\,(-1)^{j}\sum_{j=0}^{1}\int_0^b \quad Q_{[0,b]}\left[Q_{[0,b]}f_{[j]}\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)\right]\ \ dt =$$

$$= \ \sum_{j=0}^{1}\int_0^b f_{[j]}\,\mathcal{D}^{\alpha}_{\left[\frac{b}{2},b\right]}g_{[j]}(t)dt + \sum_{j=0}^{1}\int_0^b Q_{[0,b]}\left[f_{[j]}\,\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)\right]dt =$$
$$2\sum_{j=0}^{1}\int_0^b f_{[j]}\,\mathcal{D}^{\alpha}_{[b/2,b]}g_{[j]}(t)dt$$

The proof of formula (51) is analogous to the calculations presented above.

3. Reflection symmetry and localization of Euler-Lagrange equations for fractional free-motion

In paper [32] we derived Euler-Lagrange equations for an action dependent on trajectory x and its Caputo derivative of order $\alpha \in (1,2)$. For x being a real-valued function determined in interval $[0,b]$ the action looked as follows:

$$S = \int_0^b L(x(t),\ {}^{c}D^{\alpha}_{0+}x(t))dt \tag{52}$$

and after application of the minimum action principle and properties of fractional derivatives and integration, we obtained a set of equations of motion given in the theorem below.

Theorem 3.1. Let $\alpha \in (1,2)$. Then the Euler-Lagrange equations for action (52) look as follows

$$\left(\frac{\partial L}{\partial x}\right)_{[j]} + \mathcal{D}^{\alpha}_{[0,b]}\left(\frac{\partial L}{\partial\, {}^{c}D^{\alpha}_{0+}x}\right)_{[j]} - \bar{\mathcal{D}}^{\alpha}_{[0,b]}\left(\frac{\partial L}{\partial\, {}^{c}D^{\alpha}_{0+}x}\right)_{[\bar{j}]} = 0, \tag{53}$$

provided the boundary conditions are fulfilled

$$\Sigma_{[j]}(I^{2-\alpha}_{0+} + I^{2-\alpha}_{b-})\left(\frac{\partial L}{\partial c}\right)_{[j]} |_{t=0,b} < \infty$$

$$\sum_{[j]}(I^{2-\alpha}_{0+} - I^{2-\alpha}_{b-})\left(\frac{\partial L}{\partial\, {}^{c}D^{\alpha}_{0+}x}\right)_{[j]} |_{t=0,b} < \infty$$

$$\sum_{[j]}\mathcal{D}^{\alpha-1}_{[0,b]}\left(\frac{\partial L}{\partial\, {}^{c}D^{\alpha}_{0+}x}\right)_{[j]} |_{t=0,b} < \infty$$

$$\sum_{[j]}\bar{\mathcal{D}}^{\alpha-1}_{[0,b]}\left(\frac{\partial L}{\partial\, {}^{c}D^{\alpha}_{0+}x}\right)_{[j]} |_{t=0,b} < \infty\ .$$

We shall now discuss in detail the case of free motion, where the action depends solely on the derivatives:

$$S = \int_0^b \frac{1}{2}\left[\,{}^{c}D^{\alpha}_{0+}x(t)\right]^2 dt. \tag{54}$$

The Euler-Lagrange equations in interval $[0,b]$ $(j = 0,1,\ \bar{j} = 1 - j)$ look as follows:

$$\mathcal{D}^{\alpha}_{[0,b]}[\,{}^{c}D^{\alpha}_{0+}x(t)]_{[j]} - \bar{\mathcal{D}}^{\alpha}_{[0,b]}[\,{}^{c}D^{\alpha}_{0+}x(t)]_{[\bar{j}]} = 0. \tag{55}$$

Now, we calculate the [j]-components of ${}^{c}D^{\alpha}_{0+}x(t)$ denoting them as $y_{[j]}$ respectively

$$(\,{}^{c}D^{\alpha}_{0+}x)_{[0]} = {}^{c}D^{\alpha}_{[0,b]}x_{[0]} + {}^{c}\bar{D}^{\alpha}_{[0,b]}x_{[1]} = y_{[0]} \tag{56}$$

$$(\,{}^{c}D^{\alpha}_{0+}x)_{[1]} = {}^{c}D^{\alpha}_{[0,b]}x_{[1]} + {}^{c}\bar{D}^{\alpha}_{[0,b]}x_{[0]} = y_{[1]}\ . \tag{57}$$

We note that system (55) can be rewritten as the following set of four equations of order α $(j = 0,1,\ \bar{j} = 1 - j)$:

$$\mathcal{D}^{\alpha}_{[0,b]}y_{[j]} - \overline{\mathcal{D}}^{\alpha}_{[0,b]}y_{[\bar{j}]} = 0 \tag{58}$$

$$^{c}\mathcal{D}^{\alpha}_{[0,b]}x_{[j]} + \ ^{c}\overline{\mathcal{D}}^{\alpha}_{[0,b]}x_{[\bar{j}]} = y_{[j]}. \tag{59}$$

Applying the representation property given in Proposition 2.1 we obtain the following form of system (58), (59)

$$\left(1+(-1)^{j}Q_{[0,b]}\right)\mathcal{D}^{\alpha}_{[0,b/2]}y_{[j]} - \left(1-(-1)^{j}Q_{[0,b]}\right)\overline{\mathcal{D}}^{\alpha}_{[0,b/2]}y_{[\bar{j}]} = 0 \tag{60}$$

$$\left(1+(-1)^{j}Q_{[0,b]}\right)\ ^{c}\mathcal{D}^{\alpha}_{[0,b/2]}x_{[j]} + \left(1-(-1)^{j}Q_{[0,b]}\right)\ ^{c}\overline{\mathcal{D}}^{\alpha}_{[0,b/2]}x_{[\bar{j}]} = y_{[j]}. \tag{61}$$

The above system of fractional differential equations can be transformed into an equivalent system of fractional integral equations. To this aim we denote the second order derivatives of projections as

$$\frac{d^2}{dt^2}x_{[j]}(t) = z_{[j]}(t) \tag{62}$$

and derive the equivalent system in the form of

$$\left(1+(-1)^{j}Q_{[0,b]}\right)I^{2-\alpha}_{[0,b/2]}y_{[j]} - \left(1-(-1)^{j}Q_{[0,b]}\right)\overline{I}^{2-\alpha}_{[0,b/2]}y_{[\bar{j}]} = P_1(t) \tag{63}$$

$$\left(1+(-1)^{j}Q_{[0,b]}\right)I^{2-\alpha}_{[0,b/2]}z_{[j]} + \left(1-(-1)^{j}Q_{[0,b]}\right)\overline{I}^{2-\alpha}_{[0,b/2]}z_{[\bar{j}]} = y_{[j]}, \tag{64}$$

where $P_1(t) = c_1 t + c_0$ is an arbitrary polynomial of the first degree and integrals over interval $[0,b/2]$ are defined in (11), (12) for $n = 2$, $a = 0$ and b replaced by $b/2$.

Comparing systems (58), (59) and (63), (64), we observe that the derived system of integral equations is explicitly localized in interval $[0,b/2]$. Solving it in this subinterval we automatically recover the part of the trajectory in $[b/2,b]$.
We can perform a similar transformation of system (58), (59) to the integral one applying Proposition 2.2. Thanks to this representation property we can express the fractional differential operators in terms of symmetric and anti-symmetric derivatives over subinterval $[b/2,b]$

$$\left(1+(-1)^{j}Q_{[0,b]}\right)\mathcal{D}^{\alpha}_{[b/2,b]}y_{[j]} - \left(1-(-1)^{j}Q_{[0,b]}\right)\overline{\mathcal{D}}^{\alpha}_{[b/2,b]}y_{[\bar{j}]} = 0 \tag{65}$$

$$\left(1+(-1)^{j}Q_{[0,b]}\right)\ ^{c}\mathcal{D}^{\alpha}_{[b/2,b]}x_{[j]} + \left(1-(-1)^{j}Q_{[0,b]}\right)\ ^{c}\overline{\mathcal{D}}^{\alpha}_{[b/2,b]}x_{[\bar{j}]} = y_{[j]}. \tag{66}$$

Now we use formulas (11), (12) and Proposition 1.4 with $n = 2$, $a = b/2$ to obtain the system of fractional integral equations localized in subinterval $[b/2,b]$:

$$\left(1+(-1)^{j}Q_{[0,b]}\right)I^{2-\alpha}_{[b/2,b]}y_{[j]} - \left(1-(-1)^{j}Q_{[0,b]}\right)\overline{I}^{2-\alpha}_{[b/2,b]}y_{[\bar{j}]} = P_1(t) \tag{67}$$

$$\left(1+(-1)^{j}Q_{[0,b]}\right)I^{2-\alpha}_{[b/2,b]}z_{[j]} + \left(1-(-1)^{j}Q_{[0,b]}\right)\overline{I}^{2-\alpha}_{[b/2,b]}z_{[\bar{j}]} = y_{[j]}. \tag{68}$$

Conclusions

We discussed the properties of fractional derivatives connected to the reflection symmetry in a finite interval. Splitting the given function into its components with determined reflection symmetry, we obtained the representation of the symmetric and anti-symmetric fractional derivatives in a full interval as operators in a subinterval composed of certain products of the projection operators. Such a representation and the symmetry of the action integral with respect to the reflection, lead to a new formulation of variational calculus for fractional mechanics [32]. Here, we studied in detail a case of free motion and proved that Euler-Lagrange equations in this setting can be localized in subintervals.

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