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GREEN'S FUNCTION FOR VIBRATION PROBLEMS OF AN ELLIPTICAL MEMBRANE

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Abstract. The aim of the paper is to derive the Green's function of the Helmholtz operator in an elliptical region. The function is found in the form of a double series of Mathieu functions, which are obtained as a solution to the associated boundary problem. The Dirichlet condition on the boundary ellipse is assumed. The eigenvalues are the roots of characteristic equations, which are derived from the boundary condition. To construct Green's function depending on time, the orthogonality condition of the eigenfunctions in the elliptical region was used.

Introduction

The problems of membrane vibrations have been the subject of many papers [1-4]. The theoretical and numerical investigations presented in these papers concern rectangular, circular and arbitrary shaped membranes. For membranes of regular shapes (rectangle, circle), an exact solution to vibration problems can be derived. In these cases, non-homogeneous problems can be solved by using Green's function method. The problem of vibration of a non-uniform or non-regular shaped membrane is solved by using approximate (numerical) methods [5]. The method of fundamental solutions can be used as an example of such approximate methods [4]. The application of Green's function method to the vibration problem of a membrane, which occupies a finite region in the plane, requires the knowledge of Green's function for the Helmholtz equation. The method of fundamental solutions applied to the vibration problems of arbitrary shaped membranes also requires knowledge of the fundamental solution (Green's function) of the Helmholtz equation in the plane.

Green's functions of the Helmholtz equation in regular regions are well known. These functions for problems in rectangular and circular regions with various boundary conditions are given in [6]. In this paper, the derivation of Green's function for the Helmholtz equation in an elliptical region is presented. In order to separate the variables, elliptical coordinates are introduced. The solution is obtained in the form of a series of eigenfunctions of the associated boundary problem.

1. Problem formulation

The transverse vibration of a homogeneous membrane under a uniform tension is described by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} = S\nabla^2 u + f(x,y,t), \quad (x,y) \in D \tag{1}$$

where D is the domain which occupies the membrane, ∇^2 is the Laplace operator, S is the tension per unit length of membrane edge ∂D, ρ is density per unit area of the membrane, $u(x,y,t)$ is the displacement of the membrane point (x,y) at time t, $f(x,y,t)$ is the external force per unit area of the membrane acting in the perpendicular direction. Equation (1) is completed with initial and boundary conditions. We assume here zero initial conditions and the Dirichlet condition at boundary ∂D

$$u(x,y,t) = 0, \quad (x,y) \in \partial D, \ \mathrm{t} > 0 \tag{2}$$

Problem (1-2) in elliptical domain D is the subject of this paper. Boundary ∂D of the considered domain is an ellipse in a canonical form with half-axes a and b

$$\partial S: \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3}$$

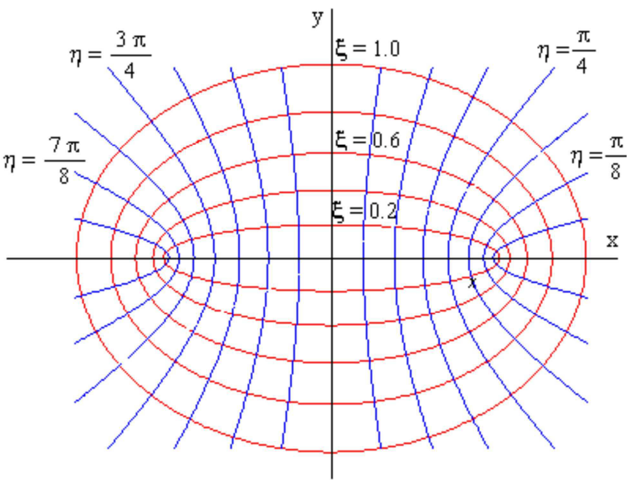

Fig. 1. Ellipses and hyperbolas in elliptical coordinates

To solve the problem, we introduce elliptical coordinates (ξ, η), which are coupled with Cartesian coordinates by the following relationships (Fig. 1):

$$\begin{cases} x = h\cosh\xi\cos\eta \\ y = h\sinh\xi\sin\eta \end{cases} \tag{4}$$

where $\xi \geq 0$, $0 \leq \eta < 2\pi$. The equation of ellipse (3) in the elliptical coordinates is: $\xi = \xi_0$, where $\xi_0 = \mathrm{artgh}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)$. The Laplace operator in elliptical coordinates has the form

$$\nabla^2 U = \frac{1}{h^2\left(\cosh^2\xi - \cos^2\eta\right)}\left(\frac{\partial^2 U}{\partial\xi^2} + \frac{\partial^2 U}{\partial\eta^2}\right) \tag{5}$$

Differential equation (1) and boundary condition (2) in the elliptical coordinates are as follows:

$$\frac{1}{h^2\left(\cosh^2\xi - \cos^2\eta\right)}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} + F(\xi,\eta,t) \tag{6}$$

$$U(\xi_0,\eta,t) = 0 \quad \text{for} \quad 0 \leq \eta < 2\pi \tag{7}$$

where $U(\xi,\eta,t) = u(x,y,t)$, $F(\xi,\eta,t) = -\frac{1}{S}f(x,y,t)$ and $c = \sqrt{S/\rho}$.

The solution of boundary problem (6), (7) can be found in the series form

$$U(\xi,\eta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Phi_{mn}(\xi,\eta)T_{mn}(t) \tag{8}$$

where functions $\Phi_{mn}(\xi,\eta)$ satisfy the homogeneous Helmholtz equation

$$\frac{1}{h^2\left(\cosh^2\xi - \cos^2\eta\right)}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\Phi_{mn} + \omega_{mn}^2\Phi_{mn} = 0 \tag{9}$$

and boundary condition

$$\Phi_{mn}(\xi_0,\eta) = 0 \quad \text{for} \quad 0 \leq \eta < 2\pi \tag{10}$$

In order to derive eigenfunctions $\Phi_{mn}(\xi,\eta)$, the method of separation of variables will be used. We assume that

$$\Phi_{mn}(\xi,\theta) = R_m(\xi,q_{mn})\Psi_m(\eta,q_{mn}) \tag{11}$$

After the substitution of (11) into equation (9) and separation of the variables, two equations are obtained:

$$\frac{d^2R_m}{d\xi^2}-\left[a-2q_{mn}\cosh 2\xi\right]R_m=0 \tag{12}$$

$$\frac{d^2\Psi_m}{d\eta^2}+\left[a-2q_{mn}\cos 2\eta\right]\Psi_m=0 \tag{13}$$

where a is the separation constant and $\omega_{mn}^2=2q_{mn}/h^2$. Taking into account boundary condition (10) and equation (11), we found that q_{mn} are the roots of the equation

$$R_m(\xi_0,q_{mn})=0 \tag{14}$$

Moreover, we assume that functions $\Psi_m(\eta,q_{mn})$ are periodic with period π or 2π. This property holds for particular values of separation constant *a*, which depends on the values of q_{mn}.

Equations (12) and (13) are well known as the radial Mathieu equation and the angular Mathieu equation respectively. The Pairs of the independent solutions of these equations are radial and angular Mathieu functions [2]:

$$R_m(\xi,q_{mn})=\begin{cases}Ce_m(\xi,q_{mn})\\ Se_{m+1}(\xi,q_{mn})\end{cases},\quad \Psi_m(\eta,q_{mn})=\begin{cases}ce_m(\eta,q_{mn})\\ se_{m+1}(\eta,q_{mn})\end{cases},\quad m=0,1,2,\ldots \tag{15}$$

Using (11) and introducing functions

$$me_{2m}(\theta,q)=ce_m(\theta,q)\qquad me_{2m+1}(\theta,q)=se_{m+1}(\theta,q),\qquad m=0,1,\ldots \tag{16}$$

$$Me_{2m}(\theta,q)=Ce_m(\theta,q)\qquad Me_{2m+1}(\theta,q)=Se_{m+1}(\theta,q),\qquad m=0,1,\ldots \tag{17}$$

function $\Phi_{mn}(\xi,\eta)$ can be written in the form

$$\Phi_{mn}(\xi,\eta)=Me_m(\xi,q_{mn})\,me_m(\eta,q_{mn}) \tag{18}$$

The angular Mathieu functions create the set of the orthogonal system in interval [0,2π], i.e. the following orthogonality condition holds:

$$\frac{1}{\pi}\int_0^{2\pi}me_m(\eta,q)me_n(\eta,q)d\eta=\delta_{mn} \tag{19}$$

where δ_{mn} is the Kronecker delta. This leads to the statement that eigenfunctions $\Phi_{mn}(\xi,\eta)$ given by (11) satisfy the condition

$$\iint_D\Phi_{mn}(\xi,\eta)\,\Phi_{kl}(\xi,\eta)d\xi\,d\eta=\pi\,\delta_{mk} \tag{20}$$

The orthogonality of the eigenfunctions will be used to determine an equation for functions $T_{mn}(t)$, which occur in series (8). First, we substitute function $U(\xi,\eta,t)$ in form (8) into differential equation (6). Next, using the standard procedure and condition (20), the following equation is obtained:

$$\ddot{T}_{mn} + c^2 \omega_{mn}^2 T_{mn} = -P_{mn}(t) \tag{21}$$

where

$$P_{mn}(t) = \frac{c^2}{\pi} \iint_D \Phi_{mn}(\xi,\eta) F(\xi,\eta,t)\, d\xi\, d\eta \tag{22}$$

The solution of equation (21), which satisfies zero initial conditions has the form

$$T_{mn}(t) = \frac{1}{c\,\omega_{mn}} \int_0^t P_{mn}(\nu) \sin c\,\omega_{mn}(t-\nu)\, d\nu \tag{23}$$

If function $F(\xi,\eta,t)$ at the right hand side of equation (6) equals [7]

$$F(\xi,\eta,t) = \frac{\delta(\xi-\zeta)\,\delta(\eta-\theta)\,\delta(t-\tau)}{h^2\left(\cosh^2\zeta - \cos^2\theta\right)} \tag{24}$$

where $\delta(\cdot)$ is the Dirac delta function, then the solution to the problem is a time dependent Green's function. Taking into account function (24) in equations (22), (23), next using equation (18) and (8) and the properties of the Dirac delta function, the Green's function for the wave equation in elliptic coordinates with a Dirichlet boundary condition at the ellipse is obtained:

$$G(\xi,\eta,t;\zeta,\theta,\tau) = \frac{1}{\pi c\, h^2\left(\cosh^2\zeta - \cos^2\theta\right)} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\omega_{mn}} \Phi_{mn}(\xi,\eta)\,\Phi_{mn}(\zeta,\theta) \sin c\omega_{mn}(t-\tau) \tag{25}$$

The assumption of function $F(\xi,\eta,t)$ occurring in the right-hand side of equation (6) in the form

$$F(\xi,\eta,t) = \frac{P\,\delta(\xi-\zeta)\,\delta(\eta-\theta)}{h^2\left(\cosh^2\zeta - \cos^2\theta\right)} e^{i\omega t} \tag{26}$$

leads to a so-called dynamic Green's function

$$G(\xi,\eta;\zeta,\theta) = \frac{1}{\pi c\, h^2\left(\cosh^2\zeta - \cos^2\theta\right)} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\omega^2 - c^2\omega_{mn}^2} \Phi_{mn}(\xi,\eta)\,\Phi_{mn}(\zeta,\theta) \tag{27}$$

Conclusions

The derivation of the Green's function of the wave equation in an elliptical region with the Dirichlet boundary condition has been presented. In order to solve the problem, elliptical coordinates were introduced. The function has the form of a double series of Mathieu functions, which are eigenfunctions of the Helmholtz operator in the considered elliptical region. Although, the solution is obtained for the Dirichlet condition at the boundary ellipse, by applying a similar approach, a Green's function satisfying the Neumann boundary condition can be derived. Using the Green's function, the solution to the non-homogenous problem of the membrane vibration can be presented in an exact form.

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