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NUMERICAL ANALYSIS OF HEAT TRANSFER IN COUNTERCURRENT BLOOD FLOW AND BIOLOGICAL TISSUE

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Abstract. Bioheat transfer in biological tissue is described by the Pennes equation, while the change of blood temperature along the artery and vein is described by ordinary differential equations, at the same time the countercurrent blood flow is taken into account. The coupling of these equations results from the boundary conditions given by the blood vessel walls. There are two methods used here in order to calculate the temperatures along the blood vessels and across biological tissue. To solve the Pennes equation, the Multiple Reciprocity Boundary Element Method (MRBEM) is applied. It should be pointed out that this method does not require discretisation of the interior of the domain. The second method used in this paper is the Finite Difference Method (FDM) and it is applied to calculate the temperatures along the blood vessels, and it complements the previous one. It is important to note that the diameter of an artery is smaller than of a vein, which results from the physiological characteristics of these blood vessels. In the final part of the paper, the results of the computations are shown and conclusions are formulated.

1. Governing equations

Biological tissue is heated by a pair of blood vessels located at the central part of the tissue cylinder, as shown in Figure 1.

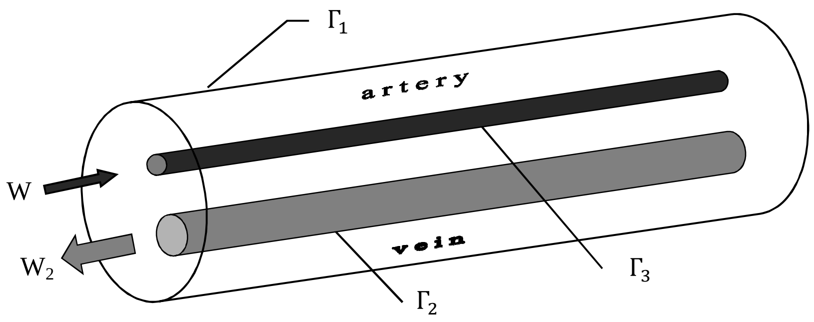

Fig. 1. Pair of blood vessels (Krogh-type tissue cylinder)

The steady state temperature field $T(x_1,x_2,z)$ in domain Ω shown in Figure 1 is described by the Pennes equation [1, 2]

$$(x_1,x_2,z)\in\Omega:\lambda\nabla^2 T(x_1,x_2,z)+G_B c_B[T_B-T(x_1,x_2,z)]+Q_{met}=0 \tag{1}$$

where λ [W/(mK)] is the tissue thermal conductivity, T_B is the blood temperature, G_B[m^{3}blood/s$\times$1/(m^{3}tissue)] is the perfusion coefficient, c_B[J/(m^{3}K)] is the specific heat of blood referred to the volume and Q_{met}[W/m^3] is the metabolic heat source. Equation (1) is supplemented by the following boundary condition on the external surface of the considered cylinder

$$(x_1,x_2,z)\in\Gamma_1:q(x_1,x_2,z)=-\lambda\mathrm{n}\cdot\nabla T(x_1,x_2,z)=0 \tag{2}$$

and the Robin boundary conditions (contact surfaces between tissue and blood vessels):

$$\begin{aligned}(x_1,x_2,z)\in\Gamma_2:q(x_1,x_2,z)=-\lambda\mathrm{n}\cdot\nabla T(x_1,x_2,z)=\alpha_2[T(x_1,x_2,z)-T_{B2}(z)]\\(x_1,x_2,z)\in\Gamma_3:q(x_1,x_2,z)=-\lambda\mathrm{n}\cdot\nabla T(x_1,x_2,z)=\alpha_3[T(x_1,x_2,z)-T_{B3}(z)]\end{aligned} \tag{3}$$

where T_{B2},T_{B3} are the blood temperatures inside of the vein and artery, respectively, α_2,α_3 are the heat transfer coefficients for the vein and artery and $\mathrm{n}=[\cos\alpha,\cos\beta]$ is the unit outward vector normal to Γ.

The second set of equations corresponding to the blood vessels temperatures are in the form of ordinary differential equations [3, 4]:

$$\frac{dT_{B2}(z)}{dz}=\frac{2\alpha_2}{w_2c_BR_2}[T_{v2}(z)-T_{B2}(z)]+\frac{Q_{Bmet}}{w_2c_B} \tag{4}$$

$$\frac{dT_{B3}(z)}{dz}=\frac{2\alpha_3}{w_3c_BR_3}[T_{v3}(z)-T_{B3}(z)]+\frac{Q_{Bmet}}{w_3c_B} \tag{5}$$

where w_2, w_3 refer to the blood velocity values in the vein and artery, T_{v2},T_{v3} are the mean temperatures on the walls of the vein and artery. Assuming the Peclet number for the network of blood vessels, we can calculate the blood velocity in the vein and artery, this means w_2, w_3.

The above equations are supplemented by initial conditions: $T_{B2}(0)=T_{B20}$ and $T_{B3}(0)=T_{B30}$, where Z is the length of the considered cylinder.

2. Solution method and idea of computation

The cross section of the considered domain is shown in Figure 2.

The multiple reciprocity boundary element method [5, 6] leads to the following integral equation corresponding to equation (1):

$$
\begin{aligned}
&B(\xi)T(\xi)+\sum_{l=0}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^{l}\int_{\Gamma} q(x)V_l^{*}(\xi,x)\mathrm{d}\Gamma = \\
&=\sum_{l=0}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^{l}\int_{\Gamma} T(x)Z_l^{*}(\xi,x)\mathrm{d}\Gamma-\frac{Q}{\lambda}\sum_{l=1}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^{l-1}\int_{\Gamma} Z_l^{*}(\xi,x)\mathrm{d}\Gamma
\end{aligned} \tag{6}
$$

where ξ is the observation point, $\xi \in (0,1]$. Coefficient $B(\xi)$ is dependent on the location of the source point. All the points located inside of Ω have $B(\xi)=1$. If point ξ belongs to any boundary Γ, then $B(\xi)=\beta/2\pi$, where β is the internal angle the boundary makes at source point ξ.

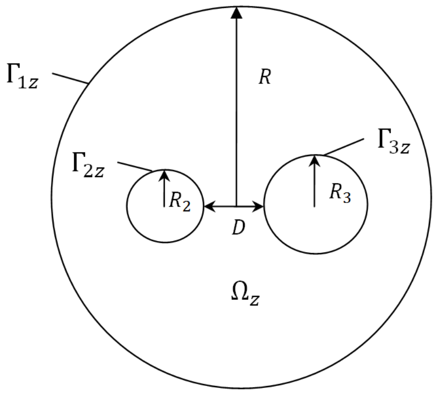

Fig. 2. Cross section of vessels

Functions $V_l^{*}(\xi,x), Z_l^{*}(\xi,x)$ are defined as follows [5, 6]:

$$V_l^{*}(\xi,x)=\frac{1}{2\pi\lambda}r^{2l}\left[A_l\ln\frac{1}{r}+B_l\right] \tag{7}$$

where r is the distance between observation point ξ and x, and

$$A_0=1,\ A_l=\frac{A_{l-1}}{4l^2},\quad l=1,2,3... \tag{8}$$

$$B_0=0,\ B_l=\frac{1}{4l^2}\left(\frac{A_{l-1}}{l}+B_{l-1}\right),\quad l=1,2,3... \tag{9}$$

$$Z_l^{*}(\xi,x)=\frac{d}{2\pi}r^{2l-2}\left(A_l-2l\left(A_l\ln\frac{1}{r}+B_l\right)\right) \tag{10}$$

where

$$d=(x_1-\xi_1)\cos\alpha_1+(x_2-\xi_2)\cos\alpha_2 \tag{11}$$

while $\cos\alpha_1, \cos\alpha_2$ are the directional cosines to outward vector n.

In order to solve equation (6), we need to perform discretisation of the boundary (c.f. Fig. 4). Let us divide boundary Γ into N elements Γ_j, $j=1,2,3...,N$. Then the integral in equation (6) can be replaced by the sums of the integrals over these elements, leading to the following equation:

$$B(\xi)T(\xi)+\sum_{l=0}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^l \sum_{j=1}^{N}\int_{\Gamma_j} q(x)V_l^*(\xi^i,x)\mathrm{d}\Gamma_j = \\ \sum_{l=0}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^l \int_{\Gamma_j}\sum_{j=1}^{N} T(x)Z_l^*(\xi^i,x)\mathrm{d}\Gamma_j - \frac{Q}{\lambda}\sum_{l=1}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^{l-1}\sum_{j=1}^{N}\int_{\Gamma_j} Z_{l+1}^*(\xi^i,x)\mathrm{d}\Gamma_j \tag{12}$$

Since the boundary elements used in this paper are of a parabolic type (c.f. Figure 3), the following is true:

$$x=(x_1,x_2)\in\Gamma_j:\begin{cases} x_1 = N_p x_1^p + N_r x_1^r + N_s x_1^s \\ x_2 = N_p x_2^p + N_r x_2^r + N_s x_2^s \end{cases} \tag{13}$$

Thus, the parabolic approximation for the temperature and heat flux is as follows:

$$x\in\Gamma_j:\begin{cases} T = N_p T_p + N_r T_r + N_s T_s \\ q = N_p q_p + N_r q_r + N_s q_s \end{cases} \tag{14}$$

where

$$N_p = \frac{\eta(1-\eta)}{2},\ N_r = (1+\eta)(1-\eta),\ N_s = \frac{\eta(1+\eta)}{2} \tag{15}$$

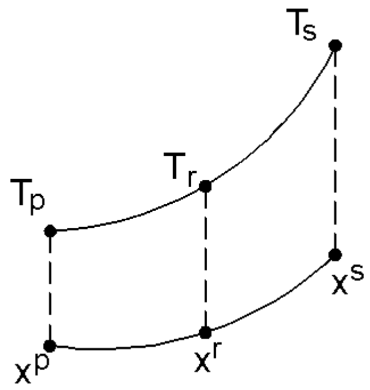

Fig. 3. Parabolic boundary element

The geometric representation of parabolic element $\mathrm{d}\Gamma_j$ can be determined and it is equal to [7]

$$\mathrm{d}\Gamma_j = \sqrt{\left(\frac{\mathrm{d}x_1}{\mathrm{d}\eta}\right)^2 + \left(\frac{\mathrm{d}x_2}{\mathrm{d}\eta}\right)^2}\,\mathrm{d}\eta = f(\eta)\mathrm{d}\eta \tag{16}$$

where:

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}\eta} &= \frac{2\eta-1}{2}x_1^p - 2\eta x_1^r + \frac{2\eta+1}{2}x_1^s \\ \frac{\mathrm{d}x_2}{\mathrm{d}\eta} &= \frac{2\eta-1}{2}x_2^p - 2\eta x_2^r + \frac{2\eta+1}{2}x_2^s \end{aligned} \tag{17}$$

Let us now make the following substitutions:

$$g_{ij}^l = \int_{\Gamma_j} V_1^*(\xi^i, x)\mathrm{d}\Gamma_j \tag{18}$$

$$h_{ij}^l = \int_{\Gamma_j} Z_l^*(\xi^i, x)\mathrm{d}\Gamma_j \tag{19}$$

For parabolic boundary elements, the integrals presented in equation (12) are equal to:

$$\int_{\Gamma_j} q(x)V_l^*(\xi^i, x)\mathrm{d}\Gamma_j = g_{ij}^{pl}q_p + g_{ij}^{rl}q_r + g_{ij}^{sl}q_s \tag{20}$$

$$\int_{\Gamma_j} T(x)Z_l^*(\xi^i, x)\mathrm{d}\Gamma_j = h_{ij}^{pl}T_p + h_{ij}^{rl}T_r + h_{ij}^{sl}T_s \tag{21}$$

$$\int_{\Gamma_j} Z_l^*(\xi^i, x)\mathrm{d}\Gamma_j = h_{ij}^{pl} + h_{ij}^{rl} + h_{ij}^{sl} \tag{22}$$

where:

$$g_{ij}^{pl} = \frac{l}{2}\int_{-1}^{1} N_p V_l^*\left(\xi_1^i, \xi_2^i, N_p x_1^p + N_r x_1^r + N_s x_1^s, N_p x_2^p + N_r x_2^r + N_s x_2^s\right)\mathrm{d}\eta \tag{23}$$

$$g_{ij}^{rl} = \frac{l}{2}\int_{-1}^{1} N_r V_l^*\left(\xi_1^i, \xi_2^i, N_p x_1^p + N_r x_1^r + N_s x_1^s, N_p x_2^p + N_r x_2^r + N_s x_2^s\right)\mathrm{d}\eta \tag{24}$$

$$g_{ij}^{sl} = \frac{l}{2}\int_{-1}^{1} N_s V_l^*\left(\xi_1^i, \xi_2^i, N_p x_1^p + N_r x_1^r + N_s x_1^s, N_p x_2^p + N_r x_2^r + N_s x_2^s\right)\mathrm{d}\eta \tag{25}$$

$$h_{ij}^{pl} = \frac{l}{2}\int_{-1}^{1} N_p Z_l^*\left(\xi_1^i, \xi_2^i, N_p x_1^p + N_r x_1^r + N_s x_1^s, N_p x_2^p + N_r x_2^r + N_s x_2^s\right)\mathrm{d}\eta \tag{26}$$

$$h_{ij}^{rl} = \frac{l}{2}\int_{-1}^{1} N_r Z_l^*\left(\xi_1^i, \xi_2^i, N_p x_1^p + N_r x_1^r + N_s x_1^s, N_p x_2^p + N_r x_2^r + N_s x_2^s\right)\mathrm{d}\eta \tag{27}$$

$$h_{ij}^{sl} = \frac{l}{2}\int_{-1}^{1} N_s Z_l^*\left(\xi_1^i, \xi_2^i, N_p x_1^p + N_r x_1^r + N_s x_1^s, N_p x_2^p + N_r x_2^r + N_s x_2^s\right)\mathrm{d}\eta \tag{28}$$

Using the above substitutes, let us rewrite the governing equation, bearing in mind that the following numeration of the boundary nodes is assumed: k = 1, 2… K. Then for i = 1, 2... K, we obtain the system of equations:

$$\begin{gathered} B_i T_i + \sum_{l=0}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^l \sum_{k=1}^{K} g_{ik}^l q_k \\ = \sum_{l=0}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^l \sum_{k=1}^{K} h_{ik}^l T_k - \frac{Q}{\lambda}\sum_{l=1}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^{l-1} \sum_{k=1}^{K} h_{ik}^l \end{gathered} \tag{29}$$

where for single nodes being the end of boundary element Γ_j and the beginning of boundary element Γ_{j+1}, we have:

$$\begin{gathered} g_{ik}^l = g_{ik}^{sl} + g_{ik+1}^{pl} \\ h_{ik}^l = h_{ik}^{sl} + h_{ik+1}^{pl} \end{gathered} \tag{30}$$

and for the central nodes of the elements:

$$\begin{gathered} g_{ik}^l = g_{ik}^{rl} \\ h_{ik}^l = h_{ik}^{rl} \end{gathered} \tag{31}$$

Using the above equations we can rewrite equation (29) as follows:

$$\sum_{k=1}^{K} G_{ik} q_k = \sum_{k=1}^{K} H_{ik} T_k + \sum_{k=1}^{K} P_{ik}, \quad i = 1, 2, 3 \ldots K \tag{32}$$

where the following substitutions have been assumed:

$$G_{ik} = \sum_{l=0}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^l g_{ik}^l \tag{33}$$

$$H_{ik} = \sum_{l=0}^{\infty}\left(\frac{G_B c_B}{\lambda}\right)^l h_{ik}^l \quad i \neq k \tag{34}$$

$$H_{ii} = \sum_{\substack{k=1 \\ k \neq i}}^{\infty} H_{ik}, \quad i = 1, 2, \dots, K \tag{35}$$

$$P_{ik} = -\frac{Q}{\lambda} \sum_{l=0}^{\infty} \left(\frac{G_B c_B}{\lambda} \right)^l h_{ik}^{l+1} \tag{36}$$

If we assume K_1 nodes are located at boundary Γ_1, K_2 nodes are located at Γ_2, and the remaining nodes K_3 are at boundary Γ_3, then we can rewrite equation (32) in following manner:

$$\begin{gathered} \sum_{k=1}^{K_1} G_{ik} q_k + \sum_{k=K_1+1}^{K_1+K_2} G_{ik} q_k + \sum_{k=K_1+K_2+1}^{K} G_{ik} q_k = \\ \sum_{k=1}^{K_1} H_{ik} T_k + \sum_{k=K_1+1}^{K_1+K_2} H_{ik} T_k + \sum_{k=K_1+K_2+1}^{K} H_{ik} T_k + \sum_{k=1}^{K} P_{ik} \end{gathered} \tag{37}$$

where i = 1, 2 … K.
We can now incorporate equations (2) and (3) into the above equation, and then we have:

$$\begin{gathered} \sum_{k=K_1+1}^{K_1+K_2} G_{ik} \alpha_2 (T_k - T_{B2}(z)) + \sum_{k=K_1+K_2+1}^{K} G_{ik} \alpha_3 (T_k - T_{B3}(z)) = \\ \sum_{k=1}^{K_1} H_{ik} T_k + \sum_{k=K_1+1}^{K_1+K_2} H_{ik} T_k + \sum_{k=K_1+K_2+1}^{K} H_{ik} T_k + \sum_{k=1}^{K} P_{ik} \end{gathered} \tag{38}$$

If we take the unknowns to the left hand side, we can rewrite the above equation in the following manner:

$$\begin{gathered} -\sum_{k=1}^{K_1} H_{ik} T_k + \sum_{k=K_1+1}^{K_1+K_2} (\alpha_2 G_{ik} - H_{ik}) T_k + \sum_{k=K_1+K_2+1}^{K} (\alpha_3 G_{ik} - H_{ik}) T_k = \\ \sum_{k=K_1+1}^{K_1+K_2} \alpha_2 G_{ik} T_{B2}(z) + \sum_{k=K_1+K_2+1}^{K} \alpha_3 G_{ik} T_{B3}(z) + \sum_{k=1}^{K} P_{ik} \end{gathered} \tag{39}$$

We can see from the equation that $T_{B2}(z), T_{B3}(z)$ needs to be known and the solution of this equation are temperatures Tk at the boundary nodes.

The next step is to determine the temperature distribution throughout the tissue, using the following equation:

$$T_i = \sum_{k=1}^{K} H_{ik} T_k - \sum_{k=1}^{K} G_{ik} q_k + \sum_{k=1}^{K} P_{ik} \tag{40}$$

The equations for blood temperature along the vessels (c.f. equations (4) and (5)) are solved using the finite difference method [4, 7]. The application of this method requires the following approximation:

$$\frac{T_{B2}(z+\Delta z) - T_{B2}}{\Delta z} = A_2[T_{v2}(z) - T_{B2}(z)] + B_2 \tag{41}$$

$$\frac{T_{B3}(z+\Delta z) - T_{B3}}{\Delta z} = -A_3[T_{v3}(z) - T_{B3}(z)] - B_3 \tag{42}$$

After modifications, we get:

$$T_{B2}(z+\Delta z) = (1 - A_2\Delta z) T_{B2}(z) + A_2 \Delta z T_{v2}(z) + B_2 \Delta z \tag{43}$$

$$T_{B3}(z+\Delta z) = (1 + A_3\Delta z) T_{B3}(z) - A_3 \Delta z T_{v3}(z) + B_3 \Delta z \tag{44}$$

where:

$$A_e = \frac{2\alpha_e}{w_e c_B R_e}, \; B = \frac{Q_{Bmet}}{w_e c_B}, \quad e = 2, 3 \tag{45}$$

$$T_{v2}(z) = \frac{1}{K_2}\sum_{i=1}^{K_2} T_i, \quad T_{v3}(z) = \frac{1}{K_3}\sum_{i=1}^{K_3} T_i \tag{46}$$

The initial temperature values for the vein and artery are known $T_{B2}(0) = T_{B20}$ and $T_{B3}(0) = T_{B30}$. We assume an arbitrary temperature at the beginning of the vein vessel (z = 0): $T_{B2}(0) = T_{B20}$.

Putting $z = 0$ in (43) and (44), we can calculate $T_{B2}(\Delta z), T_{B3}(\Delta z)$ that will feed into equation (39). From this system of equations, the boundary temperature in Γ_2, Γ_3 for $z = \Delta z$ is determined and then the mean temperatures of the vessels walls are computed using equation (46).

Having that, we can calculate $T_{B2}(2\Delta z), T_{B3}(2\Delta z)$ for cross section $2\Delta z$. This process of numerical calculations continues until z = Z. We then check the calculations of the vein blood temperature (z = 0) against boundary condition $T_{B2}(0) = T_{B20}$. If the values are not similar, we change the vein temperature for z = 0 and perform the calculations again until the two temperatures (arbitrary with the calculated one) are of similar values.

3. Entry values

The following entry data have been assumed: radius of artery $R_2 = 0.0002\,\text{m}$, radius of vein $R_3 = 0.0003\,\text{m}$, radius of tissue cylinder $R = 0.0015\,\text{m}$, distance between the two blood vessels $D = 0.0003\,\text{m}$ (see Figure 2), length of cylinder $Z = 0.05\,\text{m}$, thermal conductivity of tissue $\lambda = 0.5\,\text{W/mK}$, volumetric specific heat of blood $c_B = 3816000\,\text{J/m}^3\text{K}$, inlet arterial blood temperature is $T_{B20} = 37.5°\text{C}$, inlet venous blood temperature is $T_{B30} = 37.0°\text{C}$, perfusion coefficient is $G_B = 0.0005425\ \ 1/\text{s}$, metabolic heat sources at tissue and at vessels are $Q_{met} = 245\ \text{W/m}^3$, $Q_{Bmet} = 100\ \text{W/m}^3$, heat transfer coefficients $\alpha_2 = 5000\,\text{W/(m}^2\text{K)}$, $\alpha_3 = 3333.3\,\text{W/(m}^2\text{K)}$ and the blood velocities are $w_2 = 0.03\,\text{m/s}$, $w_3 = 0.02\,\text{m/s}$.

4. Results of computations

The boundary at each cross section has been divided into 80 parabolic boundary elements with the following node divisions: $K_1 = 120$ for boundary Γ_1, $K_2 = 24$ for boundary Γ_2, $K_3 = 36$ for boundary Γ_3. In the interior of the tissue, 240 internal nodes have been distinguished (see Fig. 4).

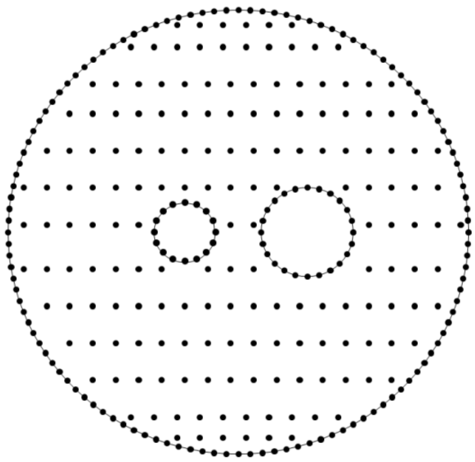

Fig. 4. Discretisation of boundary and distribution of internal nodes

The results of the computations with respect to four cross sections of temperature distribution in the tissue are plotted in Figures 5-8. In Figure 9, the changes of blood temperature along the artery-vein axes are presented.

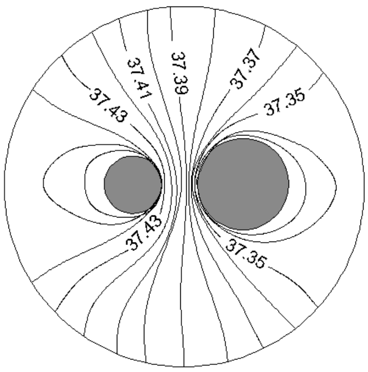

Fig. 5. Temperature distribution (z = 0 m)

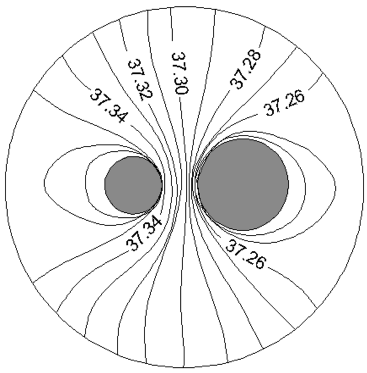

Fig. 6. Temperature distribution (z = 0.01 m)

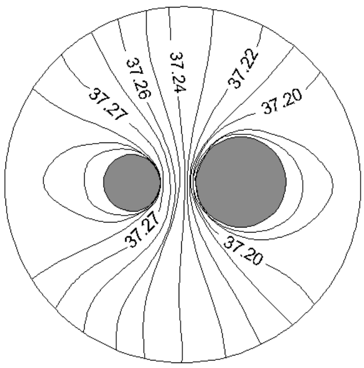

Fig. 7. Temperature distribution (z = 0.02 m)

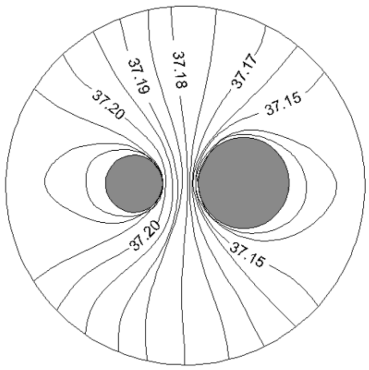

Fig. 8. Temperature distribution (z = 0.03 m)

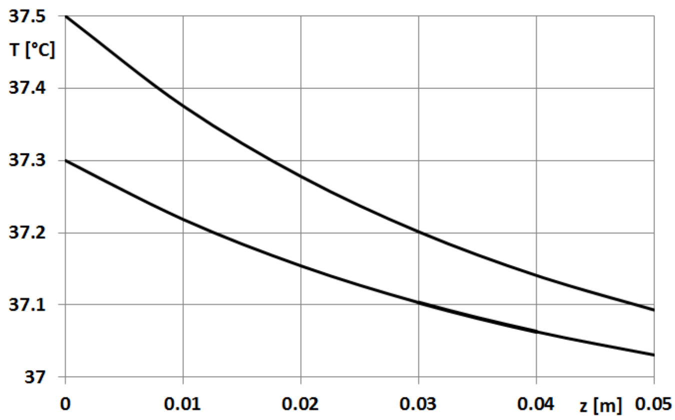

Fig. 9. Temperature distribution in artery and vein

Conclusions

A countercurrent vessels model has been established by the system of three differential equations describing the temperature field in the tissue, artery and vein. The problem has been solved by using the multiple reciprocity boundary element method (for tissue domain) and finite difference method (for vessels domain). The solution model presented here takes into account different radiuses of artery and vein. It is a very important factor in calculating temperatures as the radius of a vein is at least 2-3 times larger than that of an artery [8].

The algorithm used here to provide the solution may seem complex on the numerical computation level, but then, the mathematical model proposed here forms a good approximation of the actual thermal process in the human vessel-tissue domain. Using the algorithm proposed, it is possible to calculate different parameters describing the heat exchange between the tissue and vessels, the main one being the influence of the metabolic heat source on the thermal processes occurring in the considered domain.

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