FREQUENCY ANALYSIS OF A DOUBLE-NANOBEAM-SYSTEM

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Abstract. In this paper, a problem of transverse free vibration of a double-nanobeam-system is considered. The nanobeams of the system are coupled by an arbitrary number of translational springs. The solution of the problem by using the Green's functions properties is obtained. A numerical example is presented.

Keywords: *nanobeam system, free vibration, Green's functions*

Introduction

The vibrational behaviour of nanostructures is very important in the design of nanodevices applying in different fields of nanotechnology. The understanding of effect of small scale on vibration of the nanostructures is of great significance to the prediction of the vibrational behaviour of these nanostructures. Investigations of vibrations of the nanostructures, particularly of the nanobeams, are the subject of the papers [1-5].

Nonlocal theories for bending, buckling and vibration of nanobeams have been presented by Reddy in work [1]. The equations of motion of the nanobeam by using the nonlocal differential constitutive relations of Eringen are derived. The Euler-Bernoulli, Thimoshenko, Reddy and Levinson beam theories were considered. The paper [2] by Aydogdu is devoted to the nonlocal theories of bending, buckling and free vibrations of nanobeams. Besides the theories discussed by Reddy [1], the author presents the Aydogdu beam theory. The vibration of a nonlocal double-nanobeam-system is the subject of the paper [3] by Marmu and Adhikari. The nanobeams of the system are connected by distributed transverse springs. The presented investigation shows the small-scale effects in the free vibration of the double-nanobeam-system subjected to an initial compressive prestressed load.

The solution of the problems of free vibration of nanobeams can be obtained by applying methods such as in the classical beam theories. Exact solutions of bending, natural vibration, and buckling of simply supported beams for the considered theories were presented in the papers [1, 2]. Exact solution is also obtained to the vibration problem of nonlocal double-nanobeam-system [3] assuming that the nanobeams of the system are simply supported. Ansari et al. in the paper [4] to determine the fundamental frequencies of nanobeams used the compact difference method. In work [5] to vibration analysis of Euler-Bernoulli nanobeams the finite element method was used.

In the present paper, a solution to the free vibration problem of a system of two Euler-Bernoulli nanobeams coupled by an arbitrary number of the discrete translational springs is presented. The solution is obtained by using the Green's function method.

1. Problem formulation

A sketch of the considered system of two nanobeams connected by n-discrete translational springs is shown in Figure 1. The transverse vibrations of the nanobeams are governed by the following equations [1, 2]:

$$E_1 I_1 w_1''''\left(x_1,t\right) + N_1 w_1''\left(x_1,t\right) + \rho_1 A_1 \ddot{w}_1\left(x_1,t\right) = -\sum_{j=1}^{n} k_j \left[w_1\left(x_{1j},t\right) - w_2\left(x_{2j},t\right) \right] \delta\left(x_1 - x_{1j}\right)$$
$$+\left(e_0 a\right)^2 \sum_{j=1}^{n} k_j \left[w_1''\left(x_{1j},t\right) - w_2''\left(x_{2j},t\right) \right] \delta\left(x_1 - x_{1j}\right) + \left(e_0 a\right)^2 \left[\rho_1 A_1 \ddot{w}_1''\left(x_1,t\right) + N_1 w_1''''\left(x_1,t\right) \right] \tag{1}$$

$$E_2 I_2 w_2''''\left(x_1,t\right) + N_2 w_2''\left(x_2,t\right) + \rho_2 A_2 \ddot{w}_2\left(x_2,t\right) = \sum_{j=1}^{n} k_j \left[w_1\left(x_{1j},t\right) - w_2\left(x_{2j},t\right) \right] \delta\left(x_2 - x_{2j}\right)$$
$$-\left(e_0 a\right)^2 \sum_{i=1}^{n} k_j \left[w_1''\left(x_{1j},t\right) - w_2''\left(x_{2j},t\right) \right] \delta\left(x_2 - x_{2j}\right) + \left(e_0 a\right)^2 \left[\rho_2 A_2 \ddot{w}_2''\left(x_2,t\right) + N_2 w_2''''\left(x_1,t\right) \right] \tag{2}$$

Here w_i denotes the transverse displacement, N_i is the initial axial force, ρ_i is the mass density, E_i is the modulus of elasticity, A_i is the area of cross-section of the i-th nanobeam, $\delta(\cdot)$ denotes the Dirac delta function, x_1, x_2 are axial positions along the nanobeams, x_{1j}, x_{2j}, $\mathrm{j} = 1,2...n$ are points of the nanobeams which are joined by a j-th spring, e_0 is a constant appropriate to nanobeam material and a is an internal characteristic size. Dots ($\dot{}$) and primes ($'$) denote partial derivatives with respect to time t and position coordinate x, respectively. When $e_0 a = 0$, the equations (1)-(2) are reduced to equations of classical model of the beams system [3]. The functions $w_i(x,t)$ satisfy the boundary conditions

$$\begin{aligned} & w_i(0,t) = \frac{\partial^2 w_i}{\partial x_i^2}(0,t) = 0; \\ & w_i(L_i,t) = \frac{\partial^2 w_i}{\partial x_i^2}(L_i,t) = 0 \qquad i = 1,2 \end{aligned} \tag{3}$$

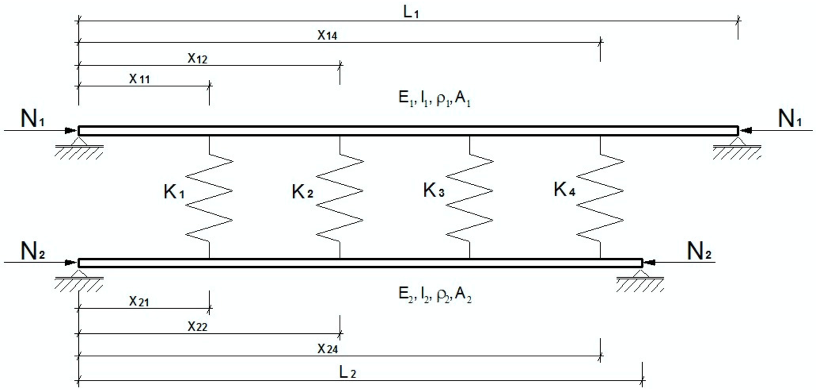

Fig. 1. A sketch of the double-nanobeam-system

2. Solution of the problem

In order to find the natural frequencies of the double-nanobeams-system, one assumes a solution of the problem in the form:

$$w_i\left(x,t\right)=\overline{W}_i\left(x\right)\cdot\cos\omega t \qquad i=1,2 \tag{4}$$

where ω is the circular frequency. Introducing new variables: $\xi_i=\dfrac{x_i}{L}$, $W_i=\dfrac{\overline{W}_i}{L_i}$ and $\overline{W}_i(x_i)=L_iW_i(\xi_i)$ into equations (1)-(2), after transformation, the following non-dimensional equations are obtained:

$$\begin{aligned}W_1''''\left(\xi_1\right)+\overline{F}_1 W_1''\!\left(\xi_1\right)-p\,\Omega^4W_1\left(\xi_1\right)=-\frac{p}{h}\sum_{j=1}^{n}K_j\left[hW_1\left(\xi_{1j}\right)-W_2\left(\xi_{2j}\right)\right]\delta\left(\xi_1-\xi_{1j}\right)\\+p\mu^2\sum_{j=1}^{n}K_j\left[W_1''\!\left(\xi_{1j}\right)-hW_2''\!\left(\xi_{2j}\right)\right]\delta\left(\xi_1-\xi_{1j}\right)\end{aligned} \tag{5}$$

$$\begin{aligned}W_2''''\left(\xi_1\right)+\overline{F}_2 W_2''\!\left(\xi_2\right)-qr^4\Omega^4W_2\left(\xi_2\right)=q\,s^2\sum_{j=1}^{n}K_j\left[hW_1\left(\xi_{1j}\right)-W_2\left(\xi_{2j}\right)\right]\delta\left(\xi_2-\xi_{2j}\right)\\-h\,q\,s^2\mu^2\sum_{j=1}^{n}K_j\left[W_1''\!\left(\xi_{1j}\right)-hW_2''\!\left(\xi_{2j}\right)\right]\delta\left(\xi_2-\xi_{2j}\right)\end{aligned} \tag{6}$$

where: $p = \frac{1}{\left(1-\mu^2 F_1\right)}$, $q = \frac{1}{1-\mu^2 h^2 F_2}$, $\Omega^4 = \frac{\rho_1 A_1 L_1^4}{E_1 I_1}\omega^2$, $\overline{F}_1 = \left(F_1 + \mu^2\Omega^4\right)p$,

$\overline{F}_2 = \left(F_2 + \mu^2 h^2 r^4 \Omega^4\right)q$, $K_j = \frac{k_j L_1^3}{E_1 I_1}$, $F_i = \frac{N_i L_i^2}{E_i I_i}$, $\mu = \frac{e_0 a}{L_1}$, $h = \frac{L_1}{L_2}$, $s^2 = \frac{E_1 I_1 L_2^3}{E_2 I_2 L_1^3}$,

$r^4 = \frac{\rho_2 A_2 E_1 I_1 L_2^4}{\rho_1 A_1 E_2 I_2 L_1^4}$.

The boundary conditions which are satisfied by the functions W_1 and W_2 follow from equations (3)

$$W_i(0) = W_i''(0) = 0; \qquad W_i(L_i) = W_i''(L_i) = 0 \qquad i = 1,2 \tag{7}$$

The solution of the boundary problem (5)-(7) can be determined by using the Green's function method [6]. The Green's functions G_i, which are necessary in this problem, satisfy the differential equation

$$\frac{\partial^4 G_i}{\partial \xi_i^4} + \overline{F}_i \frac{\partial^2 G_i}{\partial \xi_i^2} - \lambda_i^4 G_i\left(\xi_i\right) = \delta\left(\xi_i - \eta_i\right) \tag{8}$$

where $\lambda_1 = \Omega\sqrt[4]{p}$ and $\lambda_2 = r\,\Omega\,\sqrt[4]{q}$. Moreover, these functions hold the boundary conditions:

$$G_i\left(0,\eta_i\right) = \left.\frac{\partial^2 G_i}{\partial \xi_i^2}\right|_{\xi_i=0} = 0 \tag{9}$$

$$G_i\left(L_i,\eta_i\right) = \left.\frac{\partial^2 G_i}{\partial \xi_i^2}\right|_{\xi_i=L_i} = 0 \tag{10}$$

The derivation of the Green's functions is presented in section 4.

Using the properties of the Green's functions, the solution of the boundary problem (5)-(7) can be presented in the form [7, 8]:

$$\begin{aligned} W_1\left(\xi_1\right) = & -\frac{p}{h}\sum_{j=1}^{n} K_j\left[hW_1\left(\xi_{1j}\right) - W_2\left(\xi_{2j}\right)\right]G_1\left(\xi_1,\xi_{1j}\right) \\ & + p\mu^2\sum_{j=1}^{n} K_j\left[W_1''\left(\xi_{1j}\right) - hW_2''\left(\xi_{2j}\right)\right]G_1\left(\xi_1,\xi_{1j}\right) \end{aligned} \tag{11}$$

$$W_2(\xi_2) = q s^2 \sum_{j=1}^{n} K_j \left[hW_1(\xi_{1j}) - W_2(\xi_{2j}) \right] G_2(\xi_2, \xi_{2j}) - h q s^2 \mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] G_2(\xi_2, \xi_{2j}) \tag{12}$$

Substituting $\xi_1 = \xi_{1i}, \xi_2 = \xi_{2i}$, $(i = 1, 2, ...n)$ into equations (11)-(12) and in the second order derivative of the functions $W_1(\xi_1)$ and $W_2(\xi_2)$, we obtain a system of equations

$$hW_1(\xi_{1i}) = -p \sum_{j=1}^{n} K_j \left[hW_1(\xi_{1j}) - W_2(\xi_{2j}) \right] G_1(\xi_{1i}, \xi_{1j}) + h p \mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] G_1(\xi_{1i}, \xi_{1j}) \tag{13}$$

$$W_2(\xi_{2i}) = q s^2 \sum_{j=1}^{n} K_j \left[hW_1(\xi_{1j}) - W_2(\xi_{2j}) \right] G_2(\xi_{2i}, \xi_{2j}) - h q s^2 \mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] G_2(\xi_{2i}, \xi_{2j}) \tag{14}$$

$$W_1''(\xi_{1i}) = -\frac{p}{h} \sum_{j=1}^{n} K_j \left[hW_1(\xi_{1j}) - W_2(\xi_{2j}) \right] G_1''(\xi_{1i}, \xi_{1j}) + p \mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] G_1''(\xi_{1i}, \xi_{1j}) \tag{15}$$

$$hW_2''(\xi_{2i}) = q h s^2 \sum_{j=1}^{n} K_j \left[hW_1(\xi_{1j}) - W_2(\xi_{2j}) \right] G_2''(\xi_{2i}, \xi_{2j}) - q h^2 s^2 \mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] G_2''(\xi_{2i}, \xi_{2j}) \tag{16}$$

After substracting equations (15) and (16) from equations (13) and (14), respectively, we have a system

$$hW_1(\xi_{1i}) - W_2(\xi_{2i}) = -\sum_{j=1}^{n} K_j \left[hW_1(\xi_{1j}) - W_2(\xi_{2j}) \right] \left[pG_1(\xi_{1i}, \xi_{1j}) + q s^2 G_2(\xi_{2i}, \xi_{2j}) \right] + h\mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] \left[pG_1(\xi_{1i}, \xi_{1j}) + q s^2 G_2(\xi_{2i}, \xi_{2j}) \right] \tag{17}$$

$$W_1''(\xi_{1i}) - hW_2''(\xi_{2i}) = -\sum_{j=1}^{n} K_j \left[hW_1(\xi_{1j}) - W_2(\xi_{2j}) \right] \left[\frac{p}{h} G_1''(\xi_{1i},\xi_{1j}) + q\,h s^2 G_2''(\xi_{2i},\xi_{2j}) \right]$$
$$+ h\mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] \left[\frac{p}{h} G_1''(\xi_{1i},\xi_{1j}) + q\,h s^2 G_2''(\xi_{2i},\xi_{2j}) \right] \tag{18}$$

Next, substracting equation (18) from (17), we finally obtain the following system of equations:

$$hW_1(\xi_{1i}) - W_2(\xi_{2i}) - h\mu^2 \left[W_1''(\xi_{1i}) - hW_2''(\xi_{2i}) \right] =$$
$$-\sum_{j=1}^{n} K_j \left\{ \left[hW_1(\xi_{1j}) - W_2(\xi_{2j}) \right] - h\mu^2 \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] \right\} \times \tag{19}$$
$$\left\{ \left[pG_1(\xi_{1i},\xi_{1j}) + q s^2 G_2(\xi_{2i},\xi_{2j}) \right] - h\mu^2 \left[\frac{p}{h} G_1''(\xi_{1i},\xi_{1j}) + q\,h s^2 G_2''(\xi_{2i},\xi_{2j}) \right] \right\}$$

for $i = 1, 2, \ldots, n$. Assuming

$$U_i = \left[hW_1(\xi_{1i}) - W_2(\xi_{2i}) \right] - h\mu^2 \left[W_1''(\xi_{1i}) - hW_2''(\xi_{2i}) \right]$$

$$A_{ij} = \left[pG_1(\xi_{1i},\xi_{1j}) + q s^2 G_2(\xi_{2i},\xi_{2j}) \right] - \mu^2 \left[pG_1''(\xi_{1i},\xi_{1j}) + q h^2 s^2 G_2''(\xi_{2i},\xi_{2j}) \right]$$

we can write the system of equations (19) in the form

$$U_i = -\sum_{j=1}^{n} K_j U_j A_{ij} \qquad i = 1, 2, \ldots, n$$

This system of equations can be written in the matrix form

$$(\mathbf{M} + \mathrm{E}) \cdot \mathrm{U} = 0 \tag{20}$$

where: $\mathbf{M} = \left[K_j A_{ij} \right]$, $1 \le i, j \le n$, $\mathrm{U} = [U_1 \; U_2 \; U_3 \; .... \; U_n]^T$

The non-trivial solutions of equation (20) exist for these Ω, for which the determinant of the matrix $(\mathbf{M} + \mathrm{E})$ vanished. This yields the frequency equation

$$\det\left(\mathbf{M} + \mathbf{E}\right) = 0 \tag{21}$$

Equation (21) is solved numerically. The roots Ω_k, $k = 1, 2, \ldots$ of this equation are the nondimensional frequencies of the system.

3. The Green's functions determination

The Green's function $G(\xi,\eta)$, as a solution of the boundary problem (8)-(10) is found in the form [7] (index *i* is omitted)

$$G(\xi,\eta) = G_1(\xi,\eta) + G_0(\xi,\eta)\cdot H(\xi-\eta) \tag{22}$$

where $H(\xi-\eta)$ is the Heaviside function. It can be shown that both functions G_1 and G_0 satisfy the homogeneous differential equation:

$$G_0^{IV} - F_i G_0'' + \Omega^4 G_0 = 0 \tag{23}$$

Moreover the function G_0 satisfies the conditions

$$G_0\Big|_{x=\xi} = G_0'\Big|_{x=\xi} = G_0''\Big|_{x=\xi} = 0\,,\; G_0'''\Big|_{x=\xi} = 1 \tag{24}$$

The solution of the boundary problem for G_0 is

$$G_0(\xi-\eta) = \frac{-1}{\alpha^2+\beta^2}\left(\frac{1}{\alpha}\sin\alpha(\xi-\eta) - \frac{1}{\beta}\mathrm{sh}\beta(\xi-\eta)\right) \tag{25}$$

where $\alpha = \sqrt{\frac{1}{2}\left(\sqrt{F_i^2+4\Omega^4} - F_i\right)}$ and $\beta = \sqrt{\frac{1}{2}\left(\sqrt{F_i^2+4\Omega^4} + F_i\right)}$. It results that the general solution of differential equation (8) can be written in the form:

$$G(\xi-\eta) = C_1\cos\alpha\xi + C_2\sin\alpha\xi + C_3\mathrm{ch}\beta\xi + C_4\mathrm{sh}\beta\xi + G_0(\xi-\eta)\cdot H(\xi-\eta) \tag{26}$$

The constants C_1, C_2, C_3 and C_4 are determined by using boundary conditions

$$G\Big|_{\xi=0} = G_{\xi\xi}''\Big|_{\xi=0} = 0 \tag{27}$$

and

$$G\Big|_{\xi=1} = G_{\xi\xi}''\Big|_{\xi=1} = 0 \tag{28}$$

Using the boundary conditions (27) we find $C_1 = C_3 = 0$. Therefore the function $G(\xi,\eta)$ has the form

$$G(\xi-\eta) = C_2\sin\alpha\xi + C_4\mathrm{sh}\beta\xi + G_0(\xi-\eta)\cdot H(\xi-\eta) \tag{29}$$

The constants C_2 and C_4 are determined by using boundary conditions (28). Finally, the Green's function is given by equation (29) where

$$C_2 = \frac{1}{w}\left(G_0''(L-\xi) - \beta^2 G_0(L-\xi)\mathrm{sh}\beta L\right)$$

$$C_4 = \frac{-1}{w}\left(G_0''(L-\xi) - \alpha^2 G_0(L-\xi)\sin\alpha L\right)$$

$$w = (\alpha^2+\beta^2)\sin\alpha L\,\mathrm{sh}\beta L$$

4. Exemplary numerical results

The numerical results have been calculated for a system of two nanobeams of identical length and the same physical properties. These nanobeams are connected by two discrete springs and subjected to axial forces. The springs connect two points $\xi_{11} = 0.4$ and $\xi_{12} = 0.6$ of the first nanobeam to the $\xi_{21} = 0.4$ and $\xi_{22} = 0.6$ of the second nanobeam. Four different values of a nondimensional spring stiffness coefficient in computation were assumed: $K_1 = 0.1;\ 1;\ 10;\ 100$. The nanobeams are compressed by axial dimensionless force $F_1 = F_2 = 0.2$. For such a system four dimensionless natural vibration frequencies as functions of parameter μ were calculated and these are plotted in Figure 2. The computations have been performed by using the Maple package [9].

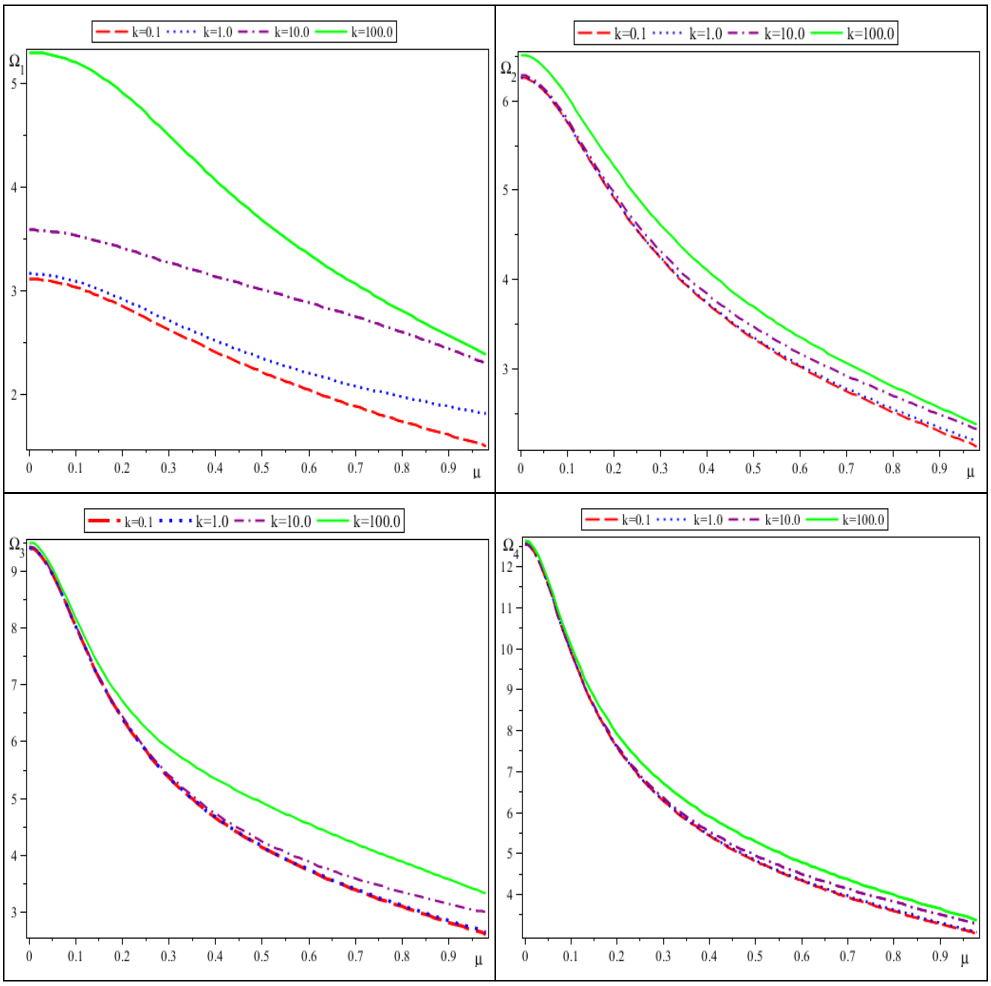

Fig. 2. The first four dimensionless natural vibration frequencies as a function of μ

The figure shows that as the parameter μ increases, the frequencies decrease for all values of spring stiffness considered. For the first frequency Ω_1 the greatest dependence of the spring stiffness is observed. When the nonlocal effects are ignored ($\mu = 0$) the above considerations revert to the classical model of the beam theory (the frequencies Ω_i, i = 1,…,4 on the Ω axis).

Conclusions

The Green function method was applied to solve the problem of transverse vibration of double-nanobeam coupled by translational springs. Simply-supported boundary conditions were employed in this study. It is observed that an increase of the parameter characterized the nanobeams (nanobeam material and internal characteristic size) causes a decrease of the frequencies of the nanobeam-system. Although the number of coupling springs considered in the presented examples was limited to two, the approach can be used to solve the problems of vibration of systems consisting of many nanobeams and coupling springs.

References

[1] Reddy J.N., Nonlocal theories for bending, buckling and vibration of beams, International Journal of Engineering Science 2007, 45, 288-307.

[2] Aydogdu M., A general nonlocal beam theory: its application to nanobeam bending, buckling and vibration, Physica E 2009, 41, 1651-1655.

[3] Murmu T., Adhikari S., Nonlocal effects in the longitudinal vibration of double-nanorod systems, Physica E 2012, 43, 415-422.

[4] Ansari R., Hisseini K., Darvizeh A., Daneshian B., A sixth-order compact finite difference method for non-classical vibration analysis of nanobeams including surface stress effects, Applied Mathematics and Computation 2013, 219, 4977-4991.

[5] Eltaher M.A., Alshorbagy A.E., Mahmoud F.F., Vibration analysis of Euler-Bernoulli nanobeams by using finite element method, Applied Mathematical Modelling 2013, 37(7), 4787-4797.

[6] Duffy D.G., Green's Functions with Applications, Chapman & Hall/CRC, 2001.

[7] Kukla S., Funkcje Greena i ich zastosowania, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2009.

[8] Ciekot A., Kukla S., Free longitudinal vibrations of nanorods system, Journal of Applied Mathematics and Computational Mechanics 2013, 2(12), 15-22.

[9] Richards D., Advanced Mathematical Methods with Maple, Cambridge University Press, 2009.