FREE VIBRATION TO A SYSTEM
OF CANTILEVER NANOBEAMS

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Abstract. This paper presents a solution to the free vibration problem of a system of two cantilever nanobeams. The nanobeams of the system are axially loaded and coupled by discrete translational springs. The solution of the problem by using the Green's function method has been obtained. A numerical example shows the effect of the small scale on free vibration frequencies of the nanobeam system considered.

Keywords: cantilever nanobeams, free vibration, Green's function method

Introduction

The classical theories of vibration of beams at macro-scales are well established, but the vibration behavior of nanostructures is far from being well understood. In the last three decades, many researchers have used the nonlocal elasticity theory to analyze the mechanical behaviors of nanostructures, particularly of nanobeams. The nonlocal theory of elasticity was initiated by Eringen [1].Various theories using nonlocal constitutive relations introduced by Eringen were the subject of the paper by Reddy [2]. In the paper are derived the equations of motion of nanobeams based on the Euler-Bernoulli, Timoshenko, Reddy and Levinson beam theories. The nonlocal beam theories for bending, buckling and vibration of nanobeams were studied by Aydogdu [3] and by Thai [4]. In these papers analytical solutions to the vibration problems of nanobeams are also presented. In the contribution by Eltaher et al. [5] the finite element method for vibration analysis of an Euler-Bernoulli nanobeam was applied. The free vibration of a cantilever nanobeam is reported in the papers by Peddieson et al. [6] and Lim et al. [7].

In this paper the free vibration problem of a system of two coupled cantilever nanobeams is studied. The buckling of a system of two nanobeams coupled by a distributed transverse spring was analyzed by Murmu and Adhikari [8]. In the present paper the subject of the consideration is the vibration of a system of two nanobeams connected by an arbitrary number of discrete translational springs. The solution of the problem was obtained using the Green's function method.

The small-scale effect on the vibration frequencies of the considered system is numerically investigated.

1. Formulation of the problem

The equations of motion for the transverse vibration of two simply-supported nanobeams that are coupled by n-discrete transitional springs were presented in paper [9]. Considering the free vibrations to a system of cantilever nanobeams we assume the transverse displacement in the form: $w_i(x,t) = \bar{W}_i(x) \cdot \cos \omega t$, $\mathrm{i} = 1, 2$, where ω is the circular frequency.
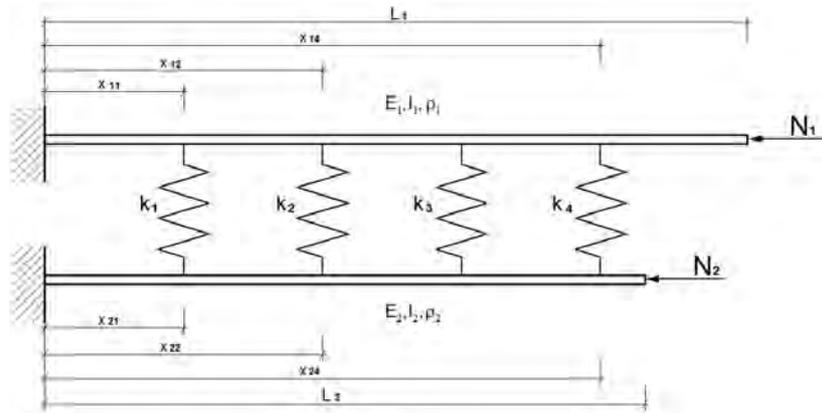

Fig. 1. A sketch of the coupled cantilever nanobeams - system

Moreover, by introducing a new variable $\xi_i = \dfrac{x_i}{L}$, $W_i = \dfrac{\bar{W}_i}{L_i}$ and $\bar{W}_i(x_i) = L_i W_i(\xi_i)$, the following system of the ordinary differential equations is obtained:

$$W_1''''(\xi_1) - \bar{F}_1 W_1''(\xi_1) - p\Omega^4 W_1(\xi_1) = p\sum_{j=1}^{n} K_j \left[W_1(\xi_{1j}) - \frac{1}{h}W_2(\xi_{2j}) \right] \delta(\xi_1 - \xi_{1j})$$

$$-p\mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] \delta(\xi_1 - \xi_{1j}) \tag{1}$$

$$W_2''''(\xi_1) - \bar{F}_2 W_2''(\xi_2) - qr^4\Omega^4 W_2(\xi_2) = -hqs^2 \sum_{j=1}^{n} K_j \left[W_1(\xi_{1j}) - \frac{1}{h}W_2(\xi_{2j}) \right] \delta(\xi_2 - \xi_{2j})$$

$$+hqs^2\mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] \delta(\xi_2 - \xi_{2j}) \tag{2}$$

where: $\quad \Omega^4 = \dfrac{\rho_1 A_1 L_1^4}{E_1 I_1}\omega^2, \quad K_j = \dfrac{k_j L_1^3}{E_1 I_1}, \quad F_i = \dfrac{N_i L_i^2}{E_i I_i}, \quad \mu = \dfrac{e_0 a}{L_1}, \quad s^2 = \dfrac{E_1 I_1 L_2^3}{E_2 I_2 L_1^3},$

$r^4 = \dfrac{\rho_2 A_2 E_1 I_1 L_2^4}{\rho_1 A_1 E_2 I_2 L_1^4}, \quad \overline{F}_1 = \dfrac{\left(F_1 + \mu^2 \Omega^4\right)}{\left(1 + \mu^2 F_1\right)}, \quad p = \dfrac{1}{\left(1 + \mu^2 F_1\right)}, \quad q = \dfrac{1}{1 + \mu^2 h^2 F_2}, \quad h = \dfrac{L_1}{L_2};$

$\overline{F}_2 = \dfrac{F_2 + \mu^2 r^4 \Omega^4 \dfrac{L_1^2}{L_2^2}}{1 + \mu^2 \dfrac{L_1^2}{L_2^2} F_2}.$

This system of equations has been solved for boundary conditions which are satisfied by the functions W_1 and W_2:

$$W_i(0) = W_i'(0) = 0; \qquad i = 1,2 \tag{3}$$

$$W_i''(L_i) = W_i'''(L_i) - \overline{F}_i W_i'\left(L_i\right) = 0 \qquad i = 1,2 \tag{4}$$

In order to solve the boundary problem (1)-(4) the Green's function method can be used. The Green's functions G_i we find as a solution of the differential equation

$$\frac{\partial^4 G_i}{\partial \xi_i^4} - \overline{F}_i \frac{\partial^2 G_i}{\partial \xi_i^2} - \lambda_i^4 G_i\left(\xi_i\right) = \delta\left(\xi_i - \eta_i\right) \tag{5}$$

where $\lambda_1 = \Omega \sqrt[4]{p}, \ \lambda_2 = r\,\Omega\sqrt[4]{q}$. The functions satisfy the homogenous boundary conditions analogues to the conditions (3)-(4):

$$G_i\left(0, \eta_i\right) = \left.\frac{\partial G_i}{\partial \xi_i}\right|_{\xi_i = 0} = 0 \tag{6}$$

$$\left.\frac{\partial^2 G_i}{\partial \xi_i^2}\right|_{\xi_i = L_i} = \left.\left(\frac{\partial^3 G_i}{\partial \xi_i^3} - \overline{F}_i \frac{\partial G_i}{\partial \xi_i}\right)\right|_{\xi_i = L_i} = 0 \tag{7}$$

Derivation of the Green's functions is presented in Section 3.

2. The solution to the problem

Using the Green's function method the solution of the problem (1)-(4) can be presented in the form:

$$W_1(\xi_1) = p\sum_{j=1}^{n} K_j \left[W_1(\xi_{1j}) - \frac{1}{h}W_2(\xi_{2j}) \right] G_1(\xi_1, \xi_{1j})$$

$$- p\mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] G_1(\xi_1, \xi_{1j}) \qquad (8)$$

$$W_2(\xi_2) = -hqs^2 \sum_{j=1}^{n} K_j \left[W_1(\xi_{1j}) - \frac{1}{h}W_2(\xi_{2j}) \right] G_2(\xi_2, \xi_{2j})$$

$$+ hqs^2\mu^2 \sum_{j=1}^{n} K_j \left[W_1''(\xi_{1j}) - hW_2''(\xi_{2j}) \right] G_2(\xi_2, \xi_{2j}) \qquad (9)$$

Substituting $\xi_1 = \xi_{1j}$, $\xi_2 = \xi_{2j}$, $(j = 1,2,...n)$ in equations (8)-(9) and in equations which express the second order derivatives of the functions $W_1(\xi_1)$ and $W_2(\xi_2)$, a system of equations is obtained. This equation system can be written in the following matrix form:

$$\mathbf{Y} \cdot \mathbf{x} = \mathbf{0} \qquad (10)$$

where

$$\mathbf{Y} = \begin{bmatrix} p\mathbf{A} - \mathbf{E} & -\dfrac{p}{h}\mathbf{A} & -p\mu^2\mathbf{A} & hp\mu^2\mathbf{A} \\[2mm] -hqs^2\mathbf{B} & qs^2\mathbf{B} - \mathbf{E} & hqs^2\mu^2\mathbf{B} & -h^2qs^2\mu^2\mathbf{B} \\[2mm] p\mathbf{C} & -\dfrac{p}{h}\mathbf{C} & -p\mu^2\mathbf{C} - \mathbf{E} & hp\mu^2\mathbf{C} \\[2mm] -hqs^2\mathbf{D} & qs^2\mathbf{D} & hqs^2\mu^2\mathbf{D} & -h^2qs^2\mu^2\mathbf{D} - \mathbf{E} \end{bmatrix}$$

$\mathbf{A} = \left[a_{ij} \right]$, $\quad \mathbf{B} = \left[b_{ij} \right]$, $\quad \mathbf{C} = \left[c_{ij} \right]$, $\quad \mathbf{D} = \left[d_{ij} \right]$, $\quad a_{ij} = K_j G_1(\xi_{1i}, \xi_{1j})$,

$b_{ij} = K_j G_2(\xi_{2i}, \xi_{2j})$, $\quad c_{ij} = K_j G_1''(\xi_{1i}, \xi_{1j})$, $\quad d_{ij} = K_j G_2''(\xi_{2i}, \xi_{2j})$, $\quad i, j = 1,2,...,n$,

$\mathbf{X} = \left[W_1(\xi_{11}) \cdots W_1(\xi_{1n}) \, W_2(\xi_{21}) \cdots W_2(\xi_{2n}) \, W_1''(\xi_{11}) \cdots W_1''(\xi_{1n}) \, W_2''(\xi_{21}) \cdots W_2''(\xi_{2n}) \right]^T$

and $\mathbf{E}$ denotes an identity matrix.

The non-trivial solutions of equation (10) exist for these values of the Ω for which the determinant of the matrix $\mathbf{Y}$ is equal to zero.

$$\det \mathbf{Y} = \mathbf{0} \qquad (11)$$

Using the properties of the determinant this equation can be also written in the form:

$$\det\begin{bmatrix} -\mathbf{E} & \mathbf{0} & \mathbf{0} & h\,p\mu^2\mathbf{A} \\ \mathbf{0} & -\mathbf{E} & \mathbf{0} & -h^2\mu^2\left(p\mathbf{A}+qs^2\mathbf{B}\right) \\ \mathbf{0} & \mathbf{0} & -\mathbf{E} & -h\left(p\mathbf{A}+qs^2\mathbf{B}-p\mu^2\mathbf{C}\right) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & p\mathbf{A}+qs^2\mathbf{B}-p\mu^2\mathbf{C}-h^2qs^2\mu^2\mathbf{D}-\mathbf{E} \end{bmatrix}=0 \qquad (12)$$

or

$$\det\left(p\mathbf{A}+qs^2\mathbf{B}-p\mu^2\mathbf{C}-h^2qs^2\mu^2\mathbf{D}-\mathbf{E}\right)=0$$

Finally we have

$$\det\left(\mathbf{M}-\mathbf{E}\right)=0 \qquad (13)$$

where $\mathbf{M}=\left[M_{ij}\right]_{1\le i,j\le n}$, and

$$M_{ij}=K_j\left[p\,G_1\left(\xi_{1i},\xi_{1j}\right)+qs^2\,G_2\left(\xi_{2i},\xi_{2j}\right)-\mu^2\left(p\,G_1{}''\left(\xi_{1i},\xi_{1j}\right)+qh^2s^2\,G_2{}''\left(\xi_{2i},\xi_{2j}\right)\right)\right]$$

The roots Ω_k ($k=1,2,\ldots$) of equation (13) are the natural frequencies of the nano-beam system and can be determined numerically.

3. Derivation of the Green's functions

The Green's functions $G_i(\xi,\eta)$ are solutions of the differential equation (5). Derivation of the Green's functions which are necessary in the considered problem is analogous to those presented in the paper [8] for simply supported nanobeams. The general solution of the equation (5) can be written in the form (index i is omitted):

$$G(\xi-\eta)=C_1\cos\alpha\xi+C_2\sin\alpha\xi+C_3\text{ch}\beta\xi+C_4\text{sh}\beta\xi+G_0(\xi-\eta)\cdot H(\xi-\eta) \quad (14)$$

where $\alpha=\sqrt{\dfrac{1}{2}\left(\sqrt{F_i^2+4\Omega^4}-F_i\right)}$, $\beta=\sqrt{\dfrac{1}{2}\left(\sqrt{F_i^2+4\Omega^4}+F_i\right)}$, $H(\cdot)$ denotes the Heaviside function and

$$G_0(u)=-\frac{1}{\alpha^2+\beta^2}\left(\frac{1}{\alpha}\sin\alpha u-\frac{1}{\beta}\sinh\beta u\right) \qquad (15)$$

The constants C_1,C_2,C_3 and C_4 occurring in equation (14) are determined using boundary conditions (6)-(7). After some transformations we obtain

$$C_1 = \frac{1}{w}(G_0''(L-\xi) \cdot \Phi'(L) - G_0'''(L-\xi) \cdot \Phi(L)) \tag{16}$$

$$C_2 = \frac{1}{w}(G_0'''(L-\xi) \cdot \Phi'(L) - \Phi'''(L) \cdot G_0''(L-\xi)) \tag{17}$$

$$C_3 = -C_1, \quad C_4 = \frac{-\alpha}{\beta} \cdot C_2 \tag{18}$$

where

$$w = \Phi'^2(L) - \Phi(L) \cdot \Phi'''(L) \tag{19}$$

$$\Phi(x) = \alpha \cdot \sin \alpha x + \beta \cdot \text{sh}\beta x \tag{20}$$

Finally, the Green's function is given by equation (14) where the function G_0, the constants C_1, C_2, C_3, C_4, w and function Φ are given by equations (15)-(20). The formulas are used in numerical calculations of the Green's function.
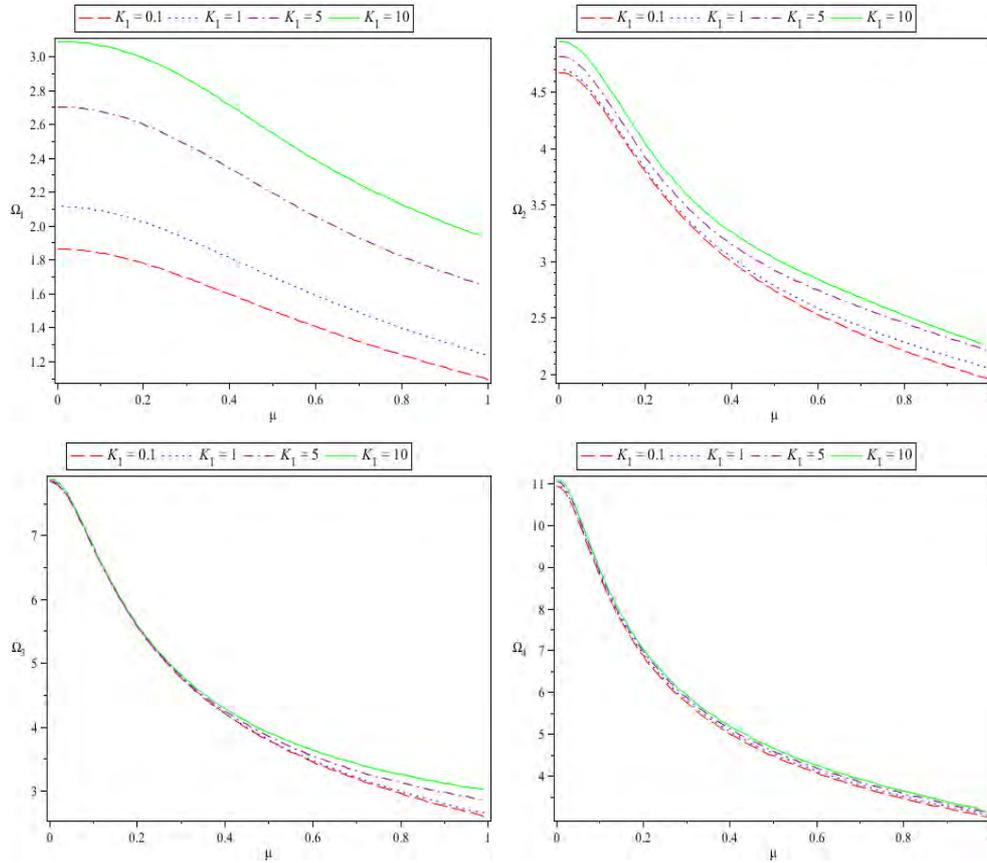

Fig. 2. The first four dimensionless natural vibration frequencies as a function of μ

4. Numerical example

The frequency equation of the system of nanobeams derived in an analytical form has been used to numerical investigation of the effect of small scale on the eigenfrequencies of this system. In computations the system of two nanobeams of identical length and the same physical properties has been assumed which were axially loaded ($F_1 = F_2 = 0.25$). The nanobeams are coupled by two translational springs characterized by the same spring stiffness coefficient. One of the springs connects the middle of the nanobeams and second connects the free nanobeam ends. The calculations were performed for four different values of spring stiffness coefficient: $K_1 = 0.1; 1.0; 5.0; 10.0$. The four dimensionless vibration frequencies as functions of parameter μ are plotted in Figure 2. The computations have been performed using the Maple package [10]. The figures show that as the parameter μ increases, the free vibration frequencies decrease for all spring stiffness coefficients considered. The spring stiffness significantly influences the first eigenfrequency Ω_1 and their effect is small for the higher frequencies.

Summary

The solution of the free translational vibration problem of the system of two cantilever nanobeams connected by discrete springs has been presented. The Green's function method was applied to obtain the eigenfrequency equation and next the method of bisection was used to numerically determine the free vibration frequencies. The effect of the small scale and the spring stiffness coefficient on the eigenfrequencies of the nanobeam system has been shown graphically. Although the number of coupling springs taken into account in the presented example was limited to two, the derived eigenequation includes the case of the system of nanobeams connected by an arbitrary number of discrete springs.

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