SYMMETRIC POLYNOMIALS IN THE 2D FOURIER EQUATION

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Abstract. The work is a continuation of the method of calculating the determinant of the block matrix in the two-dimensional case. In this paper we use the Finite Differences Method and the symmetric polynomials.

Keywords: *block matrices, symmetric polynomials*

Introduction

In this paper we return to the development of the determinant of the block matrix in the 2D case. We express this determinant by the symmetric polynomials and, consequently, by the coefficients of the output matrix (the matrix A_1).

The considered problem concerns the effective formulas expressing the symmetric polynomials of two groups of variables by the symmetric polynomials due to each of these groups.

Solution of the problem

The two-dimensional Fourier equation describing the heat flow is as follows:

$$\lambda\left(\frac{\partial^2 T(x, y, t)}{\partial x^2}+\frac{\partial^2 T(x, y, t)}{\partial y^2}\right)=\rho c\frac{\partial T(x, y, t)}{\partial t} \tag{1}$$

where λ is a thermal conductivity, c is a specific heat, ρ is a mass density and T, x, y, t denote the temperature, geometrical co-ordinates and time, respectively.

Assuming the following shape of difference quotients we get the differential approximation of the second derivatives appearing in the equation (1)

$$\frac{\Delta^2 T}{\Delta x^2} = \frac{T_{i-1,j,l} - 2T_{i,j,l} + T_{i+1,j,l}}{(\Delta x)^2}, \quad 1 \le i \le m-1$$
$$\frac{\Delta^2 T}{\Delta y^2} = \frac{T_{i,j-1,l} - 2T_{i,j,l} + T_{i,j+1,l}}{(\Delta y)^2}, \quad 1 \le j \le n-1 \tag{2}$$

and the approximation of the first derivative of the time

$$\frac{\Delta T}{\Delta t} = \frac{T_{i,j,l} - T_{i,j,l-1}}{\Delta t}, \quad 1 \le l \le q \tag{3}$$

Thus, the internal iterations taking the following differential form

$$\lambda \left(\frac{\Delta^2 T}{\Delta x^2} + \frac{\Delta^2 T}{\Delta y^2} \right) = \rho c \frac{\Delta T}{\Delta t} \tag{4}$$

and the Finite Difference Method leads to the internal system of equations

$$\begin{aligned} &\frac{\lambda}{(\Delta x)^2} T_{i-1,j,l} - \frac{2\lambda}{(\Delta x)^2} T_{i,j,l} + \frac{\lambda}{(\Delta x)^2} T_{i+1,j,l} + \\ &+ \frac{\lambda}{(\Delta y)^2} T_{i,j-1,l} - \frac{2\lambda}{(\Delta y)^2} T_{i,j,l} + \frac{\lambda}{(\Delta y)^2} T_{i,j+1,l} = \\ &= \frac{\rho c}{\Delta t} T_{i,j,l} - \frac{\rho c}{\Delta t} T_{i,j,l-1} \end{aligned} \tag{5}$$

in each time step l.
Moreover, the determinant of the matrix A_2 [1]

$$A_2 = \begin{bmatrix} A_1 & I_1 & \dots & \dots & \dots & \dots & \dots \\ I_1 & A_1 & I_1 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & I_1 & A_1 & I_1 \\ \dots & \dots & \dots & \dots & \dots & I_1 & A_1 \end{bmatrix}_{n \times n \text{ (block dimension)}} \tag{6}$$

is given by the formula

$$\det A_2 = \det \left[A_1^n - \binom{n-1}{1} A_1^{n-2} + \binom{n-2}{2} A_1^{n-4} - \dots \right] \tag{7}$$

where

$$A_1 = \begin{bmatrix} a & 1 & \dots & \dots & \dots & \dots & \dots \\ 1 & a & 1 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & 1 & a & 1 \\ \dots & \dots & \dots & \dots & \dots & 1 & a \end{bmatrix}_{m \times m} \tag{8}$$

and

$$I_1 = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 1 \end{bmatrix}_{m \times m} \tag{9}$$

We consider only the standardized case ($b_1 = 1$ in [2]) with the condition $\det A_1 \neq 0$. We apply the polynomial of degree n

$$\begin{aligned} f(x) &= x^n - \binom{n-1}{1} x^{n-2} + \binom{n-2}{2} x^{n-4} - \dots = \\ &= (x - p_1)(x - p_2)(x - p_3) \cdot \ldots \cdot (x - p_n) \end{aligned} \tag{10}$$

and then we obtain

$$\begin{aligned} \det A_2 &= \det\left[(A_1 - p_1 I_1)(A_1 - p_2 I_1) \cdot \ldots \cdot (A_1 - p_n I_1) \right] = \\ &= \underbrace{\det(A_1 - p_1 I_1)}_{W_{A_1}(p_1)} \cdot \underbrace{\det(A_1 - p_2 I_1)}_{W_{A_1}(p_2)} \cdot \ldots \cdot \underbrace{\det(A_1 - p_n I_1)}_{W_{A_1}(p_n)} = \\ &= (\lambda_1 \cdot \ldots \cdot \lambda_m)^n \left(1 - \frac{p_1}{\lambda_1}\right)\left(1 - \frac{p_1}{\lambda_2}\right) \cdot \ldots \cdot \left(1 - \frac{p_1}{\lambda_m}\right) \\ &\cdot \left(1 - \frac{p_2}{\lambda_1}\right)\left(1 - \frac{p_2}{\lambda_2}\right) \cdot \ldots \cdot \left(1 - \frac{p_2}{\lambda_m}\right) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\cdot \left(1 - \frac{p_n}{\lambda_1}\right) \cdot \ldots \cdot \left(1 - \frac{p_n}{\lambda_n}\right) = \\ &= 1 - S_1 + S_2 - \ldots + (-1)^{m \cdot n} S_{mn} \end{aligned} \tag{11}$$

where S_1 (the first symmetric polynomial of the variables indicated above) is equal to ($\tau_j = \tau_j\left(\lambda_1, \lambda_2, ..., \lambda_m\right)$ - fundamental symmetric polynomials $1 \le j \le m$)

$$S_1 = S_1\left(\frac{p_1}{\lambda_1}, \frac{p_1}{\lambda_2}, ..., \frac{p_1}{\lambda_m}; \frac{p_2}{\lambda_1}, \frac{p_2}{\lambda_2}, ..., \frac{p_2}{\lambda_m}; ...; \frac{p_n}{\lambda_1}, \frac{p_n}{\lambda_2}, ..., \frac{p_n}{\lambda_m}\right) =$$
$$= \underbrace{\left(p_1 + p_2 + ... + p_n\right)}_{\omega_1} \underbrace{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + ... + \frac{1}{\lambda_m}\right)}_{\tau_1\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_m}\right)} = \omega_1 \cdot \frac{\tau_{m-1}}{\tau_m} \tag{12}$$

while S_{mn} (the last symmetric polynomial of the indicated above variables) is of the form

$$S_{mn} = S_{mn}\left(\frac{p_1}{\lambda_1}, \frac{p_1}{\lambda_2}, ..., \frac{p_1}{\lambda_m}; \frac{p_2}{\lambda_1}, \frac{p_2}{\lambda_2}, ..., \frac{p_2}{\lambda_m}; ...; \frac{p_n}{\lambda_1}, \frac{p_n}{\lambda_2}, ..., \frac{p_n}{\lambda_m}\right) =$$
$$= \underbrace{\left(p_1 \cdot p_2 \cdot ... \cdot p_n\right)}_{\omega_n}^m \underbrace{\left(\frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2} \cdot ... \cdot \frac{1}{\lambda_m}\right)^n}_{\tau_m\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_m}\right)} = \omega_n^m \cdot \frac{1}{\tau_m^n} \tag{13}$$

The other symmetric polynomials (S_2, …, S_{mn-1}) can be calculated from the Newton formulas [3]

$$S_l = \frac{1}{l!}\det\begin{bmatrix} \delta_1 & 1 & 0 & \dots & 0 & 0 \\ \delta_2 & \delta_1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_{l-1} & \delta_{l-2} & \delta_{l-3} & \dots & \delta_1 & l-1 \\ \delta_l & \delta_{l-1} & \delta_{l-2} & \dots & \delta_2 & \delta_1 \end{bmatrix}, \quad 2 \le l \le mn-1 \tag{14}$$

determining the value of the power symmetric polynomials δ_j, $1 \le j \le l$.
We have

$$\delta_j = \delta_j\left(\frac{p_1}{\lambda_1},\frac{p_1}{\lambda_2},...,\frac{p_1}{\lambda_m};\frac{p_2}{\lambda_1},\frac{p_2}{\lambda_2},...,\frac{p_2}{\lambda_m};...;\frac{p_n}{\lambda_1},\frac{p_n}{\lambda_2},...,\frac{p_n}{\lambda_m}\right)=$$
$$=\left(\frac{p_1}{\lambda_1}\right)^j+\left(\frac{p_1}{\lambda_2}\right)^j+...+\left(\frac{p_1}{\lambda_m}\right)^j+\left(\frac{p_2}{\lambda_1}\right)^j+\left(\frac{p_2}{\lambda_2}\right)^j+...+\left(\frac{p_2}{\lambda_m}\right)^j+...+ \tag{15}$$
$$+\left(\frac{p_n}{\lambda_1}\right)^j+\left(\frac{p_n}{\lambda_2}\right)^j+...+\left(\frac{p_n}{\lambda_m}\right)^j=\underbrace{\left(p_1^j+p_2^j+...+p_n^j\right)}_{\delta_j^\omega}\underbrace{\left(\frac{1}{\lambda_1^j}+\frac{1}{\lambda_2^j}+...+\frac{1}{\lambda_m^j}\right)}_{\delta_j^\tau}$$

where $1 \le j \le mn-1$ ($\delta_1 = S_1$).

The polynomials δ_j^ω, δ_j^τ are calculated again by using the Newton formulas

$$\delta_j^\omega = \det\begin{bmatrix} \omega_1 & 1 & 0 & \dots & 0 & 0 \\ 2\omega_2 & \omega_1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (j-1)\omega_{j-1} & \omega_{j-2} & \omega_{l-3} & \dots & \omega_1 & 1 \\ j\omega_j & \omega_{j-1} & \omega_{j-2} & \dots & \omega_2 & \omega_1 \end{bmatrix}, \quad j \le \min(m,n) \tag{16}$$

($\omega_k = \omega_k\left(p_1, p_2,..., p_n\right)$ - fundamental symmetric polynomials, $1 \le k \le j$)
and so

$$\delta_j^\tau = \det\begin{bmatrix} \hat{\tau}_1 & 1 & 0 & \dots & 0 & 0 \\ 2\hat{\tau}_2 & \hat{\tau}_1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (j-1)\hat{\tau}_{j-1} & \hat{\tau}_{j-2} & \hat{\tau}_{l-3} & \dots & \hat{\tau}_1 & 1 \\ j\hat{\tau}_j & \hat{\tau}_{j-1} & \hat{\tau}_{j-2} & \dots & \hat{\tau}_2 & \hat{\tau}_1 \end{bmatrix}, \quad j \le \min(m,n) \tag{17}$$

where

$$\hat{\tau}_k = \tau_k\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},...,\frac{1}{\lambda_m}\right)=\frac{\tau_{m-k}}{\tau_m}, \quad 1 \le k \le j \tag{18}$$

Formulas (16) and (17) are true also for $\min(m,n) < j \le \max(m,n)$ and $j > \max(m,n)$. We need only the missing number of variables are assumed to be zero.

Now we return to the formula (14) to calculate the missing symmetric polynomials. Moreover, by the equation (14) it is enough to restrict considerations to the case $l \le E\left(\frac{mn-1}{2}\right)$. Indeed, the following equality takes place:

$$S_{mn-j}\left(\frac{p_1}{\lambda_1},\frac{p_1}{\lambda_2},\dots,\frac{p_1}{\lambda_m};\frac{p_2}{\lambda_1},\frac{p_2}{\lambda_2},\dots,\frac{p_2}{\lambda_m};\dots;\frac{p_n}{\lambda_1},\frac{p_n}{\lambda_2},\dots,\frac{p_n}{\lambda_m}\right)= \\ =S_j\left(\frac{\lambda_1}{p_1},\frac{\lambda_2}{p_1},\dots,\frac{\lambda_m}{p_1};\frac{\lambda_1}{p_2},\frac{\lambda_2}{p_2},\dots,\frac{\lambda_m}{p_2};\dots;\frac{\lambda_1}{p_n},\frac{\lambda_2}{p_n},\dots,\frac{\lambda_m}{p_n}\right)\cdot S_{mn} \tag{19}$$

For example

$$S_{mn-1}\left(\frac{p_1}{\lambda_1},\frac{p_1}{\lambda_2},\dots,\frac{p_1}{\lambda_m};\frac{p_2}{\lambda_1},\frac{p_2}{\lambda_2},\dots,\frac{p_2}{\lambda_m};\dots;\frac{p_n}{\lambda_1},\frac{p_n}{\lambda_2},\dots,\frac{p_n}{\lambda_m}\right)= \\ =S_1\left(\frac{\lambda_1}{p_1},\frac{\lambda_2}{p_1},\dots,\frac{\lambda_m}{p_1};\frac{\lambda_1}{p_2},\frac{\lambda_2}{p_2},\dots,\frac{\lambda_m}{p_2};\dots;\frac{\lambda_1}{p_n},\frac{\lambda_2}{p_n},\dots,\frac{\lambda_m}{p_n}\right)\cdot S_{mn}= \\ =\underbrace{\left(\lambda_1+\lambda_2+\dots+\lambda_m\right)}_{\tau_1}\underbrace{\left(\frac{1}{p_1}+\frac{1}{p_2}+\dots+\frac{1}{p_n}\right)}_{\omega_1\left(\frac{1}{p_1},\frac{1}{p_2},\dots,\frac{1}{p_n}\right)}\cdot S_{mn}= \\ =\tau_1\cdot\frac{\omega_{n-1}}{\omega_n}\cdot S_{mn}=\tau_1\cdot\frac{\omega_{n-1}}{\omega_n}\cdot\underbrace{\left(p_1\cdot p_2\cdot\dots\cdot p_n\right)^m}_{\omega_n}\underbrace{\left(\frac{1}{\lambda_1}\cdot\frac{1}{\lambda_2}\cdot\dots\cdot\frac{1}{\lambda_m}\right)^n}_{\tau_m\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},\dots,\frac{1}{\lambda_m}\right)}= \\ =\frac{\tau_1}{\tau_m^n}\cdot\omega_{n-1}\cdot\omega_n^{m-1} \tag{20}$$

(we assume that $\omega_n \neq 0$).

Now, it remains to express the fundamental polynomials τ_j and ω_k by the terms of the matrix A_1 (by the element a).

From the development of the characteristic polynomial of the matrix A_1

$$W_{A_1}(\lambda)=\det\left(A_1-\lambda I_1\right)=\det\begin{bmatrix} a-\lambda & 1 & \dots & \dots & \dots & \dots & \dots \\ 1 & a-\lambda & 1 & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & 1 & a-\lambda & 1 \\ \dots & \dots & \dots & \dots & \dots & 1 & a-\lambda \end{bmatrix}_{m\times m}=$$

$$=\left(a-\lambda\right)^n-\binom{n-1}{1}\left(a-\lambda\right)^{n-2}+\binom{n-2}{2}\left(a-\lambda\right)^{n-4}-...= \qquad (21)$$

$$=(-1)^n\left[\left(\lambda-a\right)^n-\binom{n-1}{1}\left(\lambda-a\right)^{n-2}+\binom{n-2}{2}\left(\lambda-a\right)^{n-4}-...\right]$$

we obtain the polynomials τ_j

$$\tau_j=\binom{m}{j}a^j-\binom{m-1}{1}\binom{m-2}{j-2}a^{j-2}+$$
$$+\binom{m-2}{2}\binom{m-4}{j-4}a^{j-4}-\binom{m-3}{3}\binom{m-6}{j-6}a^{j-6}+... \qquad (22)$$

By the expression (10) we get the polynomials ω_k

$$\omega_k=\begin{cases} 0 & \text{if } k \text{ is even} \\ \left(-1\right)^{\frac{k}{2}}\binom{n-\frac{k}{2}}{\frac{k}{2}} & \text{if } k \text{ is odd} \end{cases} \qquad (23)$$

So

$$\omega_2=-\binom{n-1}{1}$$
$$\omega_4=\binom{n-2}{2} \qquad (24)$$
$$\dots\dots\dots\dots\dots$$

Remark

The procedure given above constitutes an introduction to the general procedure for calculating the determinants of the matrix block in the n-dimensional case. This will be the subject of our subsequent paper.

References

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