Journal of Applied Mathematics and Computational Mechanics 2014, 13(4), 13-18

# AN APPROXIMATION OF THE FRACTIONAL INTEGRALS USING QUADRATIC INTERPOLATION

***Tomasz Błaszczyk, Jarosław Siedlecki***

*Institute of Mathematics, Czestochowa University of Technology*
*Częstochowa, Poland*
*tomasz.blaszczyk@im.pcz.pl, jaroslaw.siedlecki@im.pcz.pl*

**Abstract.** In this paper we present a numerical scheme to calculations of the left fractional integral. To calculate it we use the fractional Simpson's rule (FSR). The FSR is derived by applying quadratic interpolation. We calculate errors generated by the method for particular functions and compare the obtained results with the fractional trapezoidal rule (FTR).

***Keywords:*** *fractional integrals, quadratic interpolation, Simpson's rule*

## Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to non-integer order. The subject is as old as the differential calculus. Fractional calculus is a very useful tool in many scientific areas [1-4]. The fractional derivatives and integrals are a natural extension of the well-known integer order derivatives and integrals.

Recently, the numerical methods are used intensively and successfully to solve the fractional integral and differential equations [5-9]. However, it is still hard to develop numerical methods for some fractional equations. Bearing in mind the above-mentioned facts, many authors propose different approaches to discretization and numerical evaluation of the fractional operators [6-11].

In this paper, we propose a new approach to numerical fractional order integration. We apply methodology that is a fractional equivalent to the Simpson's rule. This method is based on quadratic interpolation.

## 1. Basic definitions

Now we will introduce the following definition and properties of fractional integration. The left Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}_+$ is defined as follows (see [12]):

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \quad \text{for } t > 0 \tag{1}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

In the further part of this paper we will use the following properties of the left Riemann-Liouville fractional integral:

$$I_{0^+}^{\alpha} \exp(t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha+k+1)} \tag{2}$$

$$I_{0^+}^{\alpha} \sin(t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{\Gamma(\alpha+2k+2)} \tag{3}$$

$$I_{0^+}^{\alpha} \cos(t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{\Gamma(\alpha+2k+1)} \tag{4}$$

## 2. Fractional Simpson's rule

Let us assume that the interval $[0, b]$ is subdivided into $N$ subintervals $[t_i, t_{i+1}]$ with constant time step $\Delta t = b/N$ by using the nodes $t_i = i\Delta t$, for $i = 0,1,\dots,N$. We wish to compute an approximation of the fractional integral (1). By the additivity of integration, we may write the left fractional integral (1) as a sum of integrals

$$I_{0^+}^{\alpha} f(t)\Big|_{t=t_i} = \frac{1}{\Gamma(\alpha)} \int_0^{t_i} \frac{f(\tau)}{(t_i-\tau)^{1-\alpha}} d\tau = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{\frac{i-2}{2}} \int_{t_{2j}}^{t_{2j+1}} \frac{f(\tau)}{(t_i-\tau)^{1-\alpha}} d\tau \tag{5}$$

Next, we replace function $f$ by the quadratic polynomial, which takes the same values as $f$ at the end points $t_{2j}$ and $t_{2j+2}$ and the midpoint $t_{2j+1}$.

$$\begin{aligned} f(\tau) \approx & \frac{(\tau - t_{2j+1})(\tau - t_{2j+2})}{2(\Delta t)^2} f(t_{2j}) - \frac{(\tau - t_{2j})(\tau - t_{2j+2})}{(\Delta t)^2} f(t_{2j+1}) \\ & + \frac{(\tau - t_{2j})(\tau - t_{2j+1})}{2(\Delta t)^2} f(t_{2j+2}) \end{aligned} \tag{6}$$

Then we denote the function values as $f_k = f(t_k)$ and put interpolation (6) into the expression (5)

$$\begin{aligned} I_{0^+}^{\alpha} f(t)\Big|_{t=t_i} \approx \frac{1}{(\Delta t)^2 \Gamma(\alpha)} \sum_{j=0}^{\frac{i-2}{2}} \Bigg[ & \int_{t_{2j}}^{t_{2j+1}} \frac{(\tau - t_{2j+1})(\tau - t_{2j+2}) f_{2j}}{2(t_i - \tau)^{1-\alpha}} d\tau \\ & - \int_{t_{2j}}^{t_{2j+1}} \frac{(\tau - t_{2j})(\tau - t_{2j+2}) f_{2j+1}}{(t_i - \tau)^{1-\alpha}} d\tau \\ & + \int_{t_{2j}}^{t_{2j+1}} \frac{(\tau - t_{2j})(\tau - t_{2j+1}) f_{2j+2}}{2(t_i - \tau)^{1-\alpha}} d\tau \Bigg] \end{aligned} \tag{7}$$

Calculating the integrals included in (7) we obtain following approximation of the left fractional integral:

$$\begin{aligned} I_{0^+}^{\alpha} f(t)\Big|_{t=t_i} & \approx \frac{(\Delta t)^{\alpha}}{2\Gamma(\alpha)} \sum_{j=0}^{\frac{i-2}{2}} \Bigg\{ \left[ f_{2j} - 2 f_{2j+1} + f_{2j+2} \right] \left[ \frac{i^2}{\alpha} c_{i,j}^{\alpha} - \frac{2i}{\alpha+1} c_{i,j}^{\alpha+1} + \frac{1}{\alpha+2} c_{i,j}^{\alpha+2} \right] \\ & - \left[ (4j+3) f_{2j} - (8j+4) f_{2j+1} + (4j+1) f_{2j+2} \right] \left[ \frac{i}{\alpha} c_{i,j}^{\alpha} - \frac{1}{\alpha+1} c_{i,j}^{\alpha+1} \right] \\ & - \left[ (2j+1)(2j+2) f_{2j} - 4j(2j+2) f_{2j+1} + 2j(2j+1) f_{2j+2} \right] \frac{i}{\alpha} c_{i,j}^{\alpha} \Bigg\} \\ & = S(t_i, \Delta t, \alpha) \end{aligned} \tag{8}$$

where $c_{i,j}^{\beta} = (i-2j)^{\beta} - (i-2j-2)^{\beta}$. When $\alpha = 1$ formula (8) takes the simplified form

$$\int_{t_0}^{t_i} f(\tau) d\tau \approx S(t_i, \Delta t, 1) = \frac{\Delta t}{3} \sum_{j=0}^{\frac{i-2}{2}} \left( f_{2j} + 4 f_{2j+1} + f_{2j+2} \right) \tag{9}$$

## 3. Results

### Example 1.

Let us consider the function $f(t) = \sin(t)$, in Tables 1 and 2 we present the approximate values of the fractional integral $I_{0^+}^{\alpha} f(t)$ at the point $t = b = 1$ and errors for different values of order $\alpha$. We compare our method - the fractional Simpson's rule (FSR), and the fractional trapezoidal rule (FTR) which was proposed by Odibat in the paper [11].

Table 1

**The numerical values and errors for integral $I_{0^+}^{0,5}\sin(t)\big|_{t=1}$**

| | FSR | | FTR | |
|---|---|---|---|---|
| $\Delta t$ | $S(t_N,\Delta t,0.5)$ | $Err(t_N,\Delta t,0.5)$ | $T(t_N,\Delta t,0.5)$ | $Err(t_N,\Delta t,0.5)$ |
| 0.1 | 0.6696793673 | $4.89\cdot10^{-6}$ | 0.6691782509 | $5.06\cdot10^{-4}$ |
| 0.05 | 0.6696838268 | $4.33\cdot10^{-7}$ | 0.6695538539 | $1.30\cdot10^{-4}$ |
| 0.0025 | 0.6696842213 | $3.83\cdot10^{-8}$ | 0.6696509827 | $3.32\cdot10^{-5}$ |
| 0.00125 | 0.6696842562 | $3.39\cdot10^{-9}$ | 0.6696758223 | $8.44\cdot10^{-6}$ |
| 0.00625 | 0.6696842593 | $3.00\cdot10^{-10}$ | 0.6696821295 | $2.13\cdot10^{-6}$ |

Table 2

**The numerical values and errors for integral $I_{0^+}^{1,5}\sin(t)\big|_{t=1}$**

| | FSR | | FTR | |
|---|---|---|---|---|
| $\Delta t$ | $S(t_N,\Delta t,0.5)$ | $Err(t_N,\Delta t,0.5)$ | $T(t_N,\Delta t,0.5)$ | $Err(t_N,\Delta t,0.5)$ |
| 0.1 | 0.2823242822 | $1.90\cdot10^{-7}$ | 0.2820860602 | $2.36\cdot10^{-4}$ |
| 0.05 | 0.2823225014 | $1.21\cdot10^{-7}$ | 0.2822634794 | $5.89\cdot10^{-5}$ |
| 0.0025 | 0.2823223880 | $7.67\cdot10^{-9}$ | 0.2823076693 | $1.47\cdot10^{-5}$ |
| 0.00125 | 0.2823223809 | $4.85\cdot10^{-10}$ | 0.2823187037 | $3.68\cdot10^{-6}$ |
| 0.00625 | 0.2823223804 | $3.05\cdot10^{-11}$ | 0.2823214613 | $9.19\cdot10^{-7}$ |

The errors were calculated using the following formula:

$$Err(t_N,\Delta t,\alpha)=\left|\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha+2k+2)}-S(t_N,\Delta t,\alpha)\right| \tag{10}$$

**Example 2.**

In this case we consider the function $f(t) = \cos(t)$. We present the approximate values of the fractional integral $I_{0^+}^{\alpha}f(t)$ at the point $t = b = 1$ and errors for different values of order $\alpha$ in Table 3.

Table 3

**The numerical values and errors for integral $I_{0^+}^{\alpha}\cos(t)\big|_{t=1}$**

| | $\alpha = 0.5$ | | $\alpha = 1.0$ | | $\alpha = 1.5$ | |
|---|---|---|---|---|---|---|
| $\Delta t$ | $S(t_N,\Delta t,\alpha)$ | $Err(t_N,\Delta t,\alpha)$ | $S(t_N,\Delta t,\alpha)$ | $Err(t_N,\Delta t,\alpha)$ | $S(t_N,\Delta t,\alpha)$ | $Err(t_N,\Delta t,\alpha)$ |
| 0.1 | 0.8460630299 | $6.24\cdot10^{-6}$ | 0.8414714528 | $4.68\cdot10^{-7}$ | 0.6696833800 | $8.79\cdot10^{-7}$ |
| 0.05 | 0.8460573780 | $5.91\cdot10^{-7}$ | 0.8414710140 | $2.92\cdot10^{-8}$ | 0.6696842001 | $5.94\cdot10^{-8}$ |
| 0.0025 | 0.8460568414 | $5.46\cdot10^{-8}$ | 0.8414709866 | $1.82\cdot10^{-9}$ | 0.6696842557 | $3.91\cdot10^{-9}$ |
| 0.00125 | 0.8460567917 | $4.98\cdot10^{-9}$ | 0.8414709849 | $1.14\cdot10^{-10}$ | 0.6696842593 | $2.52\cdot10^{-10}$ |
| 0.00625 | 0.8460567872 | $4.49\cdot10^{-10}$ | 0.8414709848 | $7.13\cdot10^{-12}$ | 0.6696842596 | $1.61\cdot10^{-11}$ |

The numerical values were computed using the FSR and the errors were calculated using the following formula

$$Err\left(t_N,\Delta t,\alpha\right)=\left|\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{\Gamma\left(\alpha+2k+1\right)}-S\left(t_N,\Delta t,\alpha\right)\right| \tag{11}$$

**Example 3.**

In the last example we consider the function $f\ (t) = \exp(t)$. We present the approximate values of the fractional integral $I_{0^+}^{\alpha} f\left(t\right)$ at the point $t=b=1$ and errors for different values of order $\alpha$ in Table 4.

Table 4

**The numerical values and errors for integral $I_{0^+}^{\alpha}\exp\left(t\right)\Big|_{t=1}$**

| | $\alpha = 0.5$ | | $\alpha = 1.0$ | | $\alpha = 1.5$ | |
|---|---|---|---|---|---|---|
| $\Delta t$ | $S(t_N,\Delta t,\alpha)$ | $Err(t_N,\Delta t,\alpha)$ | $S(t_N,\Delta t,\alpha)$ | $Err(t_N,\Delta t,\alpha)$ | $S(t_N,\Delta t,\alpha)$ | $Err(t_N,\Delta t,\alpha)$ |
| 0.1 | 2.290717870 | $1.96\cdot10^{-5}$ | 1.718282782 | $9.53\cdot10^{-7}$ | 1.162315417 | $3.67\cdot10^{-6}$ |
| 0.05 | 2.290700127 | $1.87\cdot10^{-6}$ | 1.718281888 | $5.96\cdot10^{-8}$ | 1.162318841 | $2.44\cdot10^{-7}$ |
| 0.0025 | 2.290698427 | $1.74\cdot10^{-7}$ | 1.718281832 | $3.73\cdot10^{-9}$ | 1.162319069 | $1.59\cdot10^{-8}$ |
| 0.00125 | 2.290698268 | $1.59\cdot10^{-8}$ | 1.718281829 | $2.33\cdot10^{-10}$ | 1.162319084 | $1.02\cdot10^{-9}$ |
| 0.00625 | 2.290698254 | $1.44\cdot10^{-9}$ | 1.718281828 | $1.46\cdot10^{-11}$ | 1.162319085 | $6.50\cdot10^{-11}$ |

The numerical values were computed using the FSR and the errors were calculated using the following formula

$$Err\left(t_N,\Delta t,\alpha\right)=\left|\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(\alpha+k+1\right)}-S\left(t_N,\Delta t,\alpha\right)\right| \tag{12}$$

## Conclusions

In this paper a new formula for numerical calculation of fractional integrals was presented. We derived our numerical scheme using quadratic interpolation. We compared the FSR with FTR. In comparison with FTR [11], our method (FSR) is more accurate. The approximation derived in this paper can be used directly in numerical methods for the solution of fractional order integral equations. Our results can be also extended to the right fractional integrals and to the fractional derivatives.

## References


[1] Leszczynski J.S., Blaszczyk T., Modeling the transition between stable and unstable operation while emptying a silo, Granular Matter 2011, 13, 429-438.

[2] Magin R.L., Fractional Calculus in Bioengineering, Begell House Inc., Redding 2006.

[3] Sumelka W., Blaszczyk T., Fractional continua for linear elasticity, Archives of Mechanics 2014, 66(3), 147-172.

[4] Sun H.G., Zhang Y., Chen W., Reeves D.M., Use of a variable-index fractional-derivative model to capture transient dispersion in heterogeneous media, Journal of Contaminant Hydrology 2014, 157, 47–58.

[5] Agrawal O.P., Hasan M.M., Tangpong X.W., A numerical scheme for a class of parametric problem of fractional variational calculus, J. Comput. Nonlinear Dyn. 2012, 7, 021005-1--021005-6.

[6] Baleanu D., Diethelm K., Scalas E., Trujillo J.J., Fractional Calculus Models and Numerical Methods, World Scientific, Singapore 2012.

[7] Blaszczyk T., Ciesielski M., Klimek M., Leszczynski J., Numerical solution of fractional oscillator equation, Applied Mathematics and Computation 2011, 218, 2480-2488.

[8] Blaszczyk T., Ciesielski M., Numerical solution of fractional Sturm-Liouville equation in integral form, Fract. Calc. Appl. Anal. 2014, 17, 307-320.

[9] Fu Z.J., Chen W., Ling L., Method of approximate particular solutions for constant- and variable-order fractional diffusion models, Engineering Analysis with Boundary Elements 2014, DOI: 10.1016/j.enganabound.2014.09.003.

[10] Novati P., Numerical approximation to the fractional derivative operator, Numerische Mathematik 2014, 127(3), 539-566.

[11] Odibat Z., Approximations of fractional integrals and Caputo fractional derivatives, Applied Mathematics and Computation 2006, 178, 527-533.

[12] Kilbas A.A., Srivastava H.M., Trujillo J.J., Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam 2006.