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MULTI-SERVER CLOSED QUEUEING SYSTEM WITH LIMITED BUFFER SPACE

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Abstract. In the paper, we investigate multi-server closed queueing systems with identical servers and a finite number of terminals. Requests from each terminal are characterized by a random space requirement (volume), the request service time doesn't depend on its volume and has an exponential distribution. The total requests capacity in the system is limited by a positive value (buffer space memory volume) V. For such systems, stationary requests number distribution and loss probability are determined. The analogous results for open multi-server systems are obtained as a limit case. Some numerical results are attached as well.

MSC 2010: *60K25*
Keywords: *total requests capacity, memory space, loss probability, Stieltjes convolution*

1. Introduction

Queueing systems with requests of a random space requirement (or random volume) [1, 2] are the generalization of the classical queueing models [3, 4]. They can be used to model and solve various practical problems in the design of computer and communication systems. In particular, such models can be applied for buffer space volume determination in the nodes of computer and communication networks.

The main proposition of the theory of such systems [1] is the heterogeneity of requests served by the system with respect to their space requirements or, in other words, we propose that different requests need different memory size (volume) during their presence in the system.

There are two types of queueing systems with requests of random volume [1]: 1) systems with service time independent of request volume, 2) systems, in which service time and request volume are dependent. In this paper, we analyze the closed system of the first type. Such systems characterize processes of service of a constant number of clients, who send their requests individually to a common service center containing a limited number of identical servers.

The paper is organized as follows. In Section 2, we determine the mathematical model of the analyzed system. In Section 3, we give the necessary notation, introduce the random process describing the system behavior and obtain the steady-state characteristics of the number of requests present in the system and the distribution function of the total requests volume. In Section 4, we derive the relation for requests loss probability. In Section 5, we analyze the open *M*/*M*/*n*/(*m*, *V*) system as a limit case of previous one. In Section 6, the results of some numerical calculation are presented.

2. Model description

Consider the closed system in which N terminals are served by n, $n \le N$, identical servers (see Fig. 1). Each terminal generates its request after some thinking time having an exponential distribution with parameter λ. We assume that each request needs some memory space. The size of such need (i.e. request volume) we denote by ζ and assume that ζ is a non-negative (discrete or continuous) random value (RV).

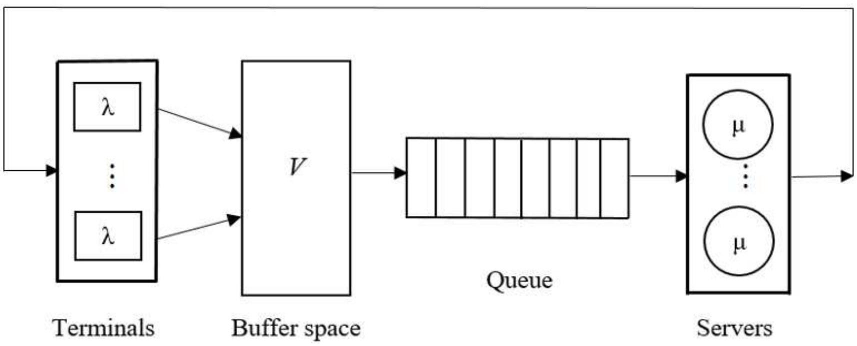

Fig. 1. Closed queueing system with bounded buffer space

Let $L(x) = P\{\zeta < x\}$ be the distribution function (DF) of the RV ζ. We denote by $\eta(t)$ the number and by $\sigma(t)$ the total volume of requests present in the system at time instant t. The values of the process $\sigma(t)$ are limited by the constant value $V > 0$ which is called the memory volume of the system. Assume that the system under consideration contains the common queue with $m \le N - n$ waiting places.

We shall denote our system by the notation *M*/*M*/*n*/*m*/(*N*,*V*).

At the epoch τ of the generation process termination, the request of volume y is accepted to the system if $\eta(\tau^-) < n + m$ and $\sigma(\tau^-) + y \le V$. Then, we have $\eta(\tau) = \eta(\tau^-) + 1$ and $\sigma(\tau) = \sigma(\tau^-) + y$. In opposite case, we have $\eta(\tau) = \eta(\tau^-)$ and

$\sigma(\tau)=\sigma(\tau^{-}),$ the request will be lost and the terminal starts the generation of the next request.

The accepted request starts its service by one of the free servers, if $\eta(\tau)\le n.$ In opposite case, the request waits for service in the queue. We assume that the order of requests service is in congruence with FIFO discipline. Service time doesn't depend on the request volume and has an exponential distribution with parameter $\mu.$ If the request of any terminal was accepted to the system, the generation of the next request starts after its service termination.

For the system under consideration, we shall determine the stationary distribution of the number of requests present in the system, the distribution function of their total volume and the loss probability of a request.

3. Determination of stationary requests number distribution and total request volume distribution function

The behavior of the system under consideration can be described by the Markov random process $(\eta(t),\sigma_1(t),...,\sigma_{\eta(t)}(t)),$ where $\sigma_j(t)$ is the volume of the jth request presenting in the system. It is clear that $\sigma(t)=\sum_{k=1}^{\eta(t)}\sigma_j(t).$

This process is characterized by the functions having the following probability sense:

$$P_k(t)=P\{\eta(t)=k\},\ k=\overline{0,n+m}; \tag{1}$$

$$G_k(y,t)=P\{\eta(t)=k,\sigma(t)<y\},\ k=\overline{1,n+m}. \tag{2}$$

It is clear that, for $k=\overline{1,n+m},$ we have $P_k(t)=G_k(V,t).$

It can be easily shown that the functions (1), (2) satisfy the following Kolmogorov-type equations:

$$\frac{dP_0(t)}{dt}=-\lambda NL(V)P_0(t)+\mu P_1(t); \tag{3}$$

$$\frac{dP_1(t)}{dt}=\lambda NL(V)P_0(t)-\lambda(N-1)\int_0^V G_1(V-y,t)\,dL(y)-\mu P_1(t)+2\mu P_2(t); \tag{4}$$

$$\begin{gathered}\frac{dP_k(t)}{dt}=\lambda(N-k+1)\int_0^V G_{k-1}(V-y,t)\,dL(y)-\lambda(N-k)\int_0^V G_k(V-y,t)\,dL(y)-\\ -k\mu P_k(t)+(k+1)\mu P_{k+1}(t),k=\overline{2,n-1};\end{gathered} \tag{5}$$

$$\frac{dP_k(t)}{dt} = \lambda(N-k+1)\int_0^V G_{k-1}(V-y,t)\,dL(y) - \lambda(N-k)\int_0^V G_k(V-y,t)\,dL(y) - \\ -n\mu P_k(t) + n\mu P_{k+1}(t),\ k=\overline{n, n+m-1}; \tag{6}$$

$$\frac{dP_{n+m}(t)}{dt} = \lambda(N-n-m+1)\int_0^V G_{n+m-1}(V-y,t)\,dL(y) - n\mu P_{n+m}(t). \tag{7}$$

In a steady state, we have obviously that $\eta(t)\Rightarrow\eta$, $\sigma(t)\Rightarrow\sigma$ when $t\to\infty$ in the sense of a weak convergence, where RV η and σ is the steady-state number of requests present in the system and their total volume, consequently. So, the following limits exist:

$$p_k = P\{\eta=k\} = \lim_{t\to\infty} P_k(t),\ k=\overline{0, n+m}; \tag{8}$$

$$g_k(y) = P\{\eta=k, \sigma<y\} = \lim_{t\to\infty} G_k(y,t),\ k=\overline{1, n+m}. \tag{9}$$

It follows from the equations (3)-(7) that the steady-state characteristics (8) and (9) satisfy the following equations:

$$0 = -\lambda N L(V)p_0 + \mu p_1; \tag{10}$$

$$0 = \lambda N L(V)p_0 - \lambda(N-1)\int_0^V g_1(V-y)\,dL(y) - \mu p_1 + 2\mu p_2; \tag{11}$$

$$0 = \lambda(N-k+1)\int_0^V g_{k-1}(V-y)\,dL(y) - \lambda(N-k)\int_0^V g_k(V-y)\,dL(y) - \\ -k\mu p_k + (k+1)\mu p_{k+1},\ k=\overline{2, n-1}; \tag{12}$$

$$0 = \lambda(N-k+1)\int_0^V g_{k-1}(V-y)\,dL(y) - \lambda(N-k)\int_0^V g_k(V-y)\,dL(y) - \\ -n\mu p_k + n\mu p_{k+1},\ k=\overline{n, n+m-1}; \tag{13}$$

$$0 = \lambda(N-n-m+1)\int_0^V g_{n+m-1}(V-y)\,dL(y) - n\mu p_{n+m}. \tag{14}$$

Introduce the following notation for Stieltjes convolution of the DF $L(x)$:

$$L_*^{(0)}(y) \equiv 1,\ \ L_*^{(k)}(y) = \int_0^y L_*^{(k-1)}(y-x)\,dL(x),\ k = 1, 2, \ldots .$$

Theorem. *The solution of the equations* (10)-(14) *has the form*

$$g_k(y) = \begin{cases} C\binom{N}{k}\rho^k L_*^{(k)}(y), & 1 \le k \le n; \\ C\dfrac{N!\rho^k}{(N-k)!n!n^{k-n}} L_*^{(k)}(y), & n+1 \le k \le n+m, \end{cases} \tag{15}$$

where C is some constant value, $\rho = \lambda/\mu$.

The theorem can be proved by direct substitution of the function (15) to equations (10)-(14). The value C can be calculated from the normalization condition $p_0 + \sum_{k=1}^{n+m} g_k(V) = 1$, whereas we have

$$C = p_0 = \left[\sum_{k=0}^{n}\binom{N}{k}\rho^k L_*^{(k)}(V) + \frac{n^n N!}{n!}\sum_{k=n+1}^{n+m}\frac{\rho^k}{n^k(N-k)!}L_*^{(k)}(V)\right]^{-1}. \tag{16}$$

Corollary. *The steady-state probabilities* p_k *have the form*

$$p_k = \begin{cases} p_0\binom{N}{k}\rho^k L_*^{(k)}(V), & 1 \le k \le n; \\ p_0\dfrac{N!\rho^k}{(N-k)!n!n^{k-n}} L_*^{(k)}(V), & n+1 \le k \le n+m, \end{cases} \tag{17}$$

where p_0 *can be obtained from the relation* (16).

Note that, for $\zeta \equiv 1$ and $V \ge n+m$, we obtain the classical closed queue *M*/*M*/*n*/*m*/*N* (see e.g. [5]).

It is clear that the steady-state distribution function $D(y)$ of the requests total volume takes the form:

$$D(y) = P\{\sigma < y\} = p_0 + \sum_{k=1}^{n+m} g_k(y) =$$

$$= p_0\left[\sum_{k=0}^{n}\binom{N}{k}\rho^k L_*^{(k)}(y) + \frac{n^n N!}{n!}\sum_{k=n+1}^{n+m}\frac{\rho^k}{n^k(N-k)!}L_*^{(k)}(y)\right],$$

where $0 < y \le V$. Of course, $D(x) = 1$, if $y > V$.

4. Loss probability

The determination of loss probability P_{loss} is based on the fact that (in a steady state) the mean number of requests accepted to the system on some time interval must be equal to the mean number of demands finishing their service on this interval. Let $\overline{\lambda}$ be the mean arrival rate of requests and K be the mean number of demands finishing their service during a unite of time. Then, we obtain $\overline{\lambda}(1-P_{loss})=K.$ For the system under consideration, we obviously have

$$\overline{\lambda}=\sum_{k=0}^{n+m}\lambda(N-k)p_k,\quad K=\sum_{k=1}^{n}k\mu p_k+n\sum_{k=n+1}^{n+m}\mu p_k,$$

whereas we obtain

$$P_{loss}=1-\frac{\sum_{k=1}^{n}kp_k+n\sum_{k=n+1}^{n+m}p_k}{\rho\sum_{k=0}^{n+m}(N-k)p_k},$$

where $\rho=\lambda/\mu.$

5. *M*/*M*/*n*/(*m*, *V*) open system

Let $N\to\infty,\ \lambda\to 0$ so that $N\lambda\to a,\ 0<a<\infty.$ In this case, we obtain an open system $M/M/n/(m,V)$ [1] with Poisson entry, where a is the rate of requests arriving.

Let $p_k^*,\ k=\overline{0,n+m},$ be the steady-state probability of the presence of k requests in the open system and P_{loss}^* be the steady-state loss probability for this system. Relations for probabilities p_k^* can be obtained by passing to limits in the relations (17), when $N\to\infty.$ Hence, for p_k^* determination, it is essential to calculate the limit $\lim\limits_{N\to\infty}\frac{N!\rho^k}{(N-k)!}=\lim\limits_{N\to\infty}\frac{N!}{(N-k)!}\left(\frac{\lambda}{\mu}\right)^k.$

Using Stirling asymptotic formula $l!\sim\sqrt{2\pi l}\,l^l e^{-l}$ and introducing the notation $s=N-k,$ we get:

$$\lim_{N\to\infty}\frac{N!}{(N-k)!}\left(\frac{\lambda}{\mu}\right)^k=\lim_{s\to\infty}\left(1+\frac{k}{s}\right)^s\cdot e^k\cdot\frac{a^k}{\mu^k}=(n\rho^*)^k,$$

where $\rho^*=a/(n\mu).$ As a result, we have the following known relations [1]:

$$p_0^*=\left[\sum_{k=0}^{n}\frac{(n\rho^*)^k}{k!}L_*^{(k)}(V)+\frac{n^n}{n!}\sum_{k=n+1}^{n+m}(\rho^*)^k L_*^{(k)}(V)\right]^{-1},$$

$$p_k^* = \begin{cases} p_0^* \dfrac{(n\rho^*)^k}{k!} L_*^{(k)}(V), & 1 \le k \le n; \\ p_0^* \dfrac{n^n (\rho^*)^k}{n!} L_*^{(k)}(V), & n+1 \le k \le n+m. \end{cases}$$

For the loss probability P_{loss}^*, we obtain

$$P_{loss}^* = 1 - \frac{1}{n\rho^*} \sum_{k=1}^{n-1} k p_k^* - \frac{1}{\rho^*} \left(1 - \sum_{k=0}^{n-1} p_k^* \right).$$

6. Numerical examples

Note that Laplace-Stieltjes convolutions calculation is generally not very simple. But it is possible for some special cases. For example, if the request volume has an exponential distribution with parameter μ, we have for $y > 0$ that

$$L_*^{(k)}(y) = 1 - e^{-fV} \sum_{j=0}^{k-1} \frac{(fy)^j}{j!}, \quad k = 0, 1, \dots .$$

For the case of gamma-distribution with parameters $\alpha > 0$, $f > 0$, when

$$L(y) = L(\alpha, f, y) = \frac{1}{\Gamma(\alpha)} \int_0^y f^{\alpha} u^{\alpha - 1} e^{-fu} du, x > 0,$$

we have $L_*^{(k)}(y) = L(k\alpha, f, y)$, $k = 1, 2, \dots$.

For the case of uniform on $[a; b]$, $a \ge 0$, distribution, we obtain [6]:

$$L_*^{(k)}(y) = \left(\frac{-1}{b-a} \right)^k \sum_{j=0}^{k} \frac{(-1)^j [(b-a)j - bk + y]^k H((b-a)j - bk + y)}{j!(k-j)!}, \quad k = 1, 2, \dots,$$

where $H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases}$

On Figures 2, 3 and 4, we present the dependence of loss probability P_{loss} on the value of buffer space V. In all examples we put $\lambda = 1, \mu = 1, N = 3, n = 2, m = 0$ (line 1) or $m = 1$ (line 2). Figure 2 presents the case of exponential distribution of request volume with parameter $f = 1$. Figure 3 presents the case of its uniform distribution with parameters $a = 0, b = 2$. In Figure 4 we present the case of gamma-distribution with parameters $\alpha = 2, f = 2$.

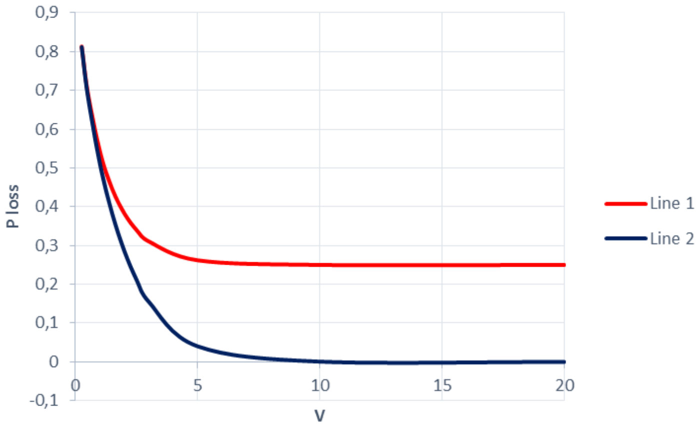

Fig. 2. Loss probability for the case of exponential distribution of the request volume

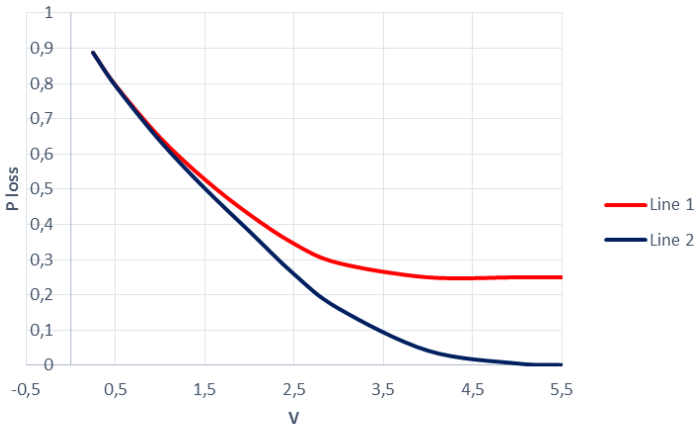

Fig. 3. Loss probability for the case of uniform distribution of the request volume

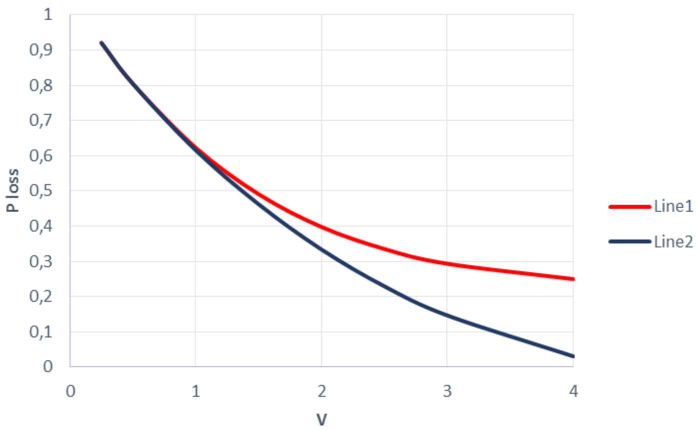

Fig. 4. Loss probability for the case of gamma distribution of the request volume

7. Conclusions

In the paper, we investigate the closed queueing systems with random volume requests. We show that, in such investigation, it is possible to take into account the heterogeneity of requests having different space requirements. We obtain the steady-state distribution of the requests number present in the system, the distribution function of the requests total volume and loss probability. We also show that the open Erlang system is the limit case of the investigated one. Some simple numerical examples are attached as well.

References

[1] Tikhonenko O., Probability Methods of Information System Analysis, Akademicka Oficyna Wydawnicza EXIT, Warsaw 2006 (in Polish).

[2] Tikhonenko O.M., Queueing Models in Information Systems, Universitrtskoe, Minsk 1990 (in Russian).

[3] Bocharov P.P., D'Apice C., Pechinkin A.V., Salerno S., Queueing Theory, VSP, Utrecht-Boston 2004.

[4] Gnedenko B. V., Kovalenko I. N., Introduction to Queueing Theory, Birkhäuser, Boston 1989.

[5] Breuer L., Baum D., An Introduction to Queueing Theory and Matrix-Analytic Methods, Springer, 2005.

[6] Feller W., An Introduction to Probability Theory and its Applications. Vol. 2. Wiley, New York 1971.