

UNIFORMLY BOUNDED NEMYTSKIJ OPERATORS ACTING BETWEEN THE BANACH SPACES OF GENERALIZED HÖLDER FUNCTIONS

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Abstract. We investigate the Nemytskij (composition, superposition) operators acting between Banach spaces of r -times differentiable functions defined on the closed intervals of the real line with the r -derivatives satisfying a generalized Hölder condition. The main result says that if such a Nemytskij operator is uniformly bounded (in a special case uniformly continuous) then its generator is an affine function with respect to the second variable, i.e., the Matkowski representation holds. This extends an earlier result where an operator is assumed to be Lipschitzian.

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1. Introduction

The Nemytskij operators occur frequently in the theory of integral equations, differential equations, variational calculus or in optimization theory. There is vast literature devoted to applications of the results connected with the theory of the composition operators (see, for instance [1], where such operators occur in the problems of option pricing within the Black-Scholes model for time-dependent volatility).

Let $I = [a, b]$ be an interval of the real line \mathbb{R} ($a, b \in \mathbb{R}, a < b$) and let \mathbb{R}^I denote the set of all functions $\varphi: I \rightarrow \mathbb{R}$. For a given function $h: I \times \mathbb{R} \rightarrow \mathbb{R}$, the mapping $H: \mathbb{R}^I \rightarrow \mathbb{R}^I$ defined by

$$H(\varphi)(x) = h(x, \varphi(x)), \quad \varphi \in \mathbb{R}^I, \quad x \in I,$$

is called a composition (Nemytskij or superposition) operator of a generator h .

In 1982 J. Matkowski [2] proved that if H maps a Banach space $(Lip[0,1], \|\cdot\|_{Lip[0,1]})$ of Lipschitzian functions $\varphi \in \mathbb{R}^{[0,1]}$ with the classical $Lip[0,1]$ -norm into itself and is globally Lipschitzian, i.e., if for some constant $c \geq 0$,

$$\|H(\varphi_1) - H(\varphi_2)\|_{Lip[0,1]} \leq c\|\varphi_1 - \varphi_2\|_{Lip[0,1]}, \quad \varphi_1, \varphi_2 \in Lip[0,1],$$

then

$$h(x, y) = \alpha(x)y + \beta(x), \quad x \in [0,1], \quad y \in \mathbb{R}, \quad (1)$$

for some functions $\alpha, \beta \in Lip[0,1]$, i.e., h is an affine function with respect to the second variable. Analogous results for representations of the generators of globally Lipschitzian operators have been proved for some other function spaces [3-7]. Later, it was observed that these results remain true if the Lipschitz norm-continuity of H is replaced by its uniform continuity [8, 9].

In 2011, J. Matkowski [10] proved, under very general assumptions, that for the function spaces including the Hölder spaces as a special case, the uniform continuity of the operator H can be replaced by a much weaker condition of the uniform boundedness (which is weaker than norm-boundedness), see also [11, 12].

The purpose of this paper is to show that if H maps the space $W_\gamma^n[a, b]$ of n -times differentiable functions with the n -derivative satisfying a generalized Hölder condition into $W_\gamma^r[a, b]$, where $n \geq r$, and H is uniformly bounded, then the Matkowski representation holds, i.e., the generator h of H must be of the form (1) for some $\alpha, \beta \in W_\gamma^n[a, b]$. This extends the main result of [5, 6] where H is assumed to be Lipschitzian.

2. Preliminaries

Given a closed interval $[a, b] \subset \mathbb{R}$ ($a, b \in \mathbb{R}$, $a < b$) put $d := b - a$. Let $\gamma: [0, d] \rightarrow [0, \infty)$ satisfy the following condition:

- (I) γ is increasing and concave, $\gamma(0) = 0$, $\lim_{t \rightarrow 0^+} \gamma(t) = \gamma(0)$,
 $\lim_{t \rightarrow d^-} \gamma(t) = \gamma(d)$.

Given $r \in \mathbb{N}$, denote by $W_\gamma^r[a, b]$ the set of all r -times differentiable functions $\varphi: [a, b] \rightarrow \mathbb{R}$ such that their r -th derivatives satisfy the following condition (so called *the generalized Hölder condition* or the γ -Hölder condition): there exists a constant $M \geq 0$ such that

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \leq M\gamma(|x - \bar{x}|), \quad \bar{x}, x \in [a, b], \quad (2)$$

where $\gamma: [0, d] \rightarrow [0, \infty)$ satisfy condition (I).

It is known that the space $W_\gamma^r[a, b]$ endowed with the norm

$$\|\varphi\|_{W_\gamma^r[a, b]} := \sum_{k=0}^r |\varphi^{(k)}(a)| + \sup \left\{ \frac{|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})|}{\gamma(|x - \bar{x}|)}; x, \bar{x} \in [a, b], x \neq \bar{x} \right\} \quad (3)$$

is a Banach space [3, 4].

Let $LipC^r[a, b]$, $r \in \mathbb{N}$, stand for the class of all r -times differentiable functions $\varphi: [a, b] \rightarrow \mathbb{R}$, whose r -th derivatives satisfy the Lipschitz condition on $[a, b]$.

Remark 2.1. Observe, that if $\varphi \in LipC^r[a, b]$ with the constant L_r , then φ fulfils (2) with the constant $\frac{L_r d}{\gamma(d)}$.

Indeed, by (Γ) , we get the monotonicity of the function

$$(0, d] \ni t \rightarrow \frac{t}{\gamma(t)} \in (0, \infty),$$

and, consequently, we have

$$|\varphi^{(r)}(x) - \varphi^{(r)}(\bar{x})| \leq L_r |x - \bar{x}| \leq L_r \frac{|x - \bar{x}|}{\gamma(|x - \bar{x}|)} \gamma(|x - \bar{x}|) \leq L_r \frac{d}{\gamma(d)} \gamma(|x - \bar{x}|),$$

for all $x, \bar{x} \in [a, b]$.

Remark 2.2 ([6], Remark 1). If $\varphi \in W_\gamma^r[a, b]$ and $\gamma'_+(0)$ is finite, where $\gamma'_+(0)$ denotes the right derivative of γ at $t = 0$, then $LipC^r[a, b] = W_\gamma^r[a, b]$.

Remark 2.3. Let us notice that the function defined by $\gamma(t) = t^\alpha$, where $0 < \alpha < 1$, $t \in [0, d]$, fulfils the assumption (Γ) and $\gamma'_+(0) = +\infty$.

Definition 2.1. Let $X \subset \mathbb{R}$ and a function $h: X \times \mathbb{R} \rightarrow \mathbb{R}$ be fixed. The mapping $H: \mathbb{R}^X \rightarrow \mathbb{R}^X$ given by

$$H(\varphi)(x) = h(x, \varphi(x)), \quad \varphi \in \mathbb{R}^X, \quad x \in X,$$

is said to be a composition (Nemytskij or superposition) operator. The function h is referred to as the generator of the operator H .

Assume now, that the function h fulfils the following condition:

(i) $h: [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is r -times differentiable in $[a, b] \times \mathbb{R}$, $r \in \mathbb{N}$, and all its r -th order partial derivatives fulfill the generalized Hölder conditions with respect to the first variable and Lipschitz condition with respect to the second variable.

Define, by recurrence, functions $h_k: [a, b] \times \mathbb{R}^{k+1} \rightarrow \mathbb{R}$, $k = 0, 1, \dots, r$, as follows:

$$\begin{cases} h_0(x, y_0) := h(x, y_0) \\ h_{k+1}(x, y_0, \dots, y_{k+1}) := \frac{\partial h_k}{\partial x} + \frac{\partial h_k}{\partial y_0} y_1 + \dots + \frac{\partial h_k}{\partial y_k} y_{k+1}. \end{cases} \quad (4)$$

Before formulating the main results of our paper, we will need the following two lemmas.

Lemma 2.1 ([6], Lemma 1). If the assumption (i) is fulfilled, then the functions h_k , $k = 0, 1, \dots, r$, defined by (4), are $(r - k)$ -times continuously differentiable in $[a, b] \times \mathbb{R}^{k+1}$ and, for $k \geq 2$, we have

$$h_{k+1}(x, y_0, \dots, y_k) = p_k(x, y_0, y_1) + q_k(x, y_0, \dots, y_{k-1}) + \frac{\partial h}{\partial y}(x, y_0) y_k$$

where

$$p_k(x, y_0, y_1) = \sum_{i=0}^k \binom{k}{i} \frac{\partial^k h}{\partial x^{k-i} \partial y_0^i}(x, y_0) y_1^i, \quad (5)$$

and $q_k(x, y_0, \dots, y_{k-1})$ is a polynomial of the variables y_1, \dots, y_{k-1} whose coefficients are functions of the variables (x, y_0) and are of the class C^{r-k+1} with respect to x and y_0 .

Lemma 2.2 ([6], Lemma 2). If the assumption (i) is fulfilled, $\varphi \in W_\gamma^r[a, b]$ and

$$\psi(x) := h(x, \varphi(x)), \quad x \in [a, b],$$

then the derivatives of ψ fulfil the following system of equations:

$$\psi^{(k)}(x) = h_k(x, \varphi(x), \dots, \varphi^{(k)}(x)), \quad k = 0, 1, \dots, r,$$

and $\psi \in W_\gamma^r[a, b]$.

3. Main results

Theorem 3.1. Let $a, b \in \mathbb{R}$, $n, r \in \mathbb{N}$, $a < b$, $n \geq r$, be fixed and let a function $h: [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be such that for any $x \in [a, b]$ the function $h(x, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ is r -times differentiable and its r -th derivatives satisfy the Lipschitz condition on \mathbb{R} . If the composition operator H of the generator h maps the space $W_\gamma^n[a, b]$ into $W_\gamma^r[a, b]$ and satisfies the inequality

$$\|H(\varphi_1) - H(\varphi_2)\|_{W_\gamma^r[a,b]} \leq f\left(\|\varphi_1 - \varphi_2\|_{W_\gamma^n[a,b]}\right), \quad \varphi_1, \varphi_2 \in W_\gamma^n[a,b], \quad (6)$$

for some function $f: [0, \infty) \rightarrow [0, \infty)$, then there exist $\alpha \in W_\gamma^r[a,b]$ and $\beta \in W_\gamma^r[a,b]$ such that

$$h(x, y) = \alpha(x)y + \beta(x), \quad x \in [a, b], \quad y \in \mathbb{R}.$$

Proof. For every fixed $y \in \mathbb{R}$ the function $\varphi(t) = y$, $t \in [a, b]$, as a constant, is n -times differentiable and its n -th derivatives satisfy the generalized Hölder condition, so, by assumption, $H(\varphi) = h(\cdot, y) \in W_\gamma^r[a,b]$. Thus, taking into account the assumption, it follows that the function h fulfils (i).

Let us fix $x, \bar{x} \in [a, b]$, $x < \bar{x}$, and define the function

$$\psi_0(t) := \begin{cases} 0 & \text{if } a \leq t < x \\ (\bar{x} - x)^{r-n-1}(t - x) & \text{if } x \leq t \leq \bar{x} \\ (\bar{x} - x)^{r-n} & \text{if } \bar{x} \leq b. \end{cases}$$

It is obvious that $\psi_0 \in Lip[a,b]$ with the constant $L_0 = (\bar{x} - x)^{r-n-1}$, so, by Remark 1, ψ_0 satisfy the generalized Hölder condition with the constant

$$M = (\bar{x} - x)^{r-n-1} \frac{d}{\gamma(d)}.$$

Let us take arbitrary $z_1, z_2 \in \mathbb{R}$. Putting

$$\varphi_i(t) = z_i \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} \psi_0(s) ds, \quad i = 1, 2, \quad t \in [a, b] \quad (7)$$

we get

$$\varphi_i^{(k)}(t) = \frac{1}{(n-k-1)!} \int_a^t (t-s)^{n-k-1} \psi_0(s) ds, \quad k = 1, \dots, n-1, \quad (8)$$

and

$$\varphi_i^{(n)}(t) = \psi_0(t), \quad (9)$$

for all $t \in [a, b]$, $i = 1, 2$.

Hence, $\varphi_i \in W_\gamma^n[a,b]$, $i = 1, 2$, and, by the definition of the function ψ_0 , for all $t \in [a, x]$, we obtain

$$\varphi_i(t) = z_i, \quad i = 1, 2, \quad (10)$$

and

$$\varphi_i^{(k)}(t) = 0, \quad k = 1, \dots, n, \quad i = 1, 2, \quad (11)$$

Since, by (8),

$$\varphi_i^{(k)}(\bar{x}) = \frac{1}{(n-k-1)!} \int_x^{\bar{x}} (t-s)^{n-k-1} \psi_0(s) ds = \frac{1}{(n-k+1)!} (\bar{x}-x)^{r-k},$$

$$k = 1, \dots, n-1,$$

and, by (9),

$$\varphi_i^{(n)}(\bar{x}) = (\bar{x}-x)^{r-n},$$

therefore

$$\lim_{\bar{x} \rightarrow x} \varphi_i^{(k)}(\bar{x}) = \begin{cases} 0 & \text{if } k = 1, \dots, r-1 \\ \frac{1}{(n-k+1)!} & \text{if } k = r \end{cases}, \quad i = 1, 2. \quad (12)$$

Setting $u_r := p_r + q_r$, where p_r, q_r are defined in Lemma 1, inequality (6) has the following form

$$\begin{aligned} & \sum_{k=0}^r \left| h_k(a, \varphi_1(a), \dots, \varphi_1^{(k)}(a)) - h_k(a, \varphi_2(a), \dots, \varphi_2^{(k)}(a)) \right| + \\ \sup_{t \neq \bar{t}} & \left\{ \left| u_r(t, \varphi_1(t), \dots, \varphi_1^{(r-1)}(t)) - u_r(\bar{t}, \varphi_1(\bar{t}), \dots, \varphi_1^{(r-1)}(\bar{t})) - u_r(t, \varphi_2(t), \dots, \varphi_2^{(r-1)}(t)) \right. \right. \\ & \left. \left. + u_r(\bar{t}, \varphi_2(\bar{t}), \dots, \varphi_2^{(r-1)}(\bar{t})) + \frac{\partial h}{\partial y}(t, \varphi_1(t)) \varphi_1^{(r)}(t) - \frac{\partial h}{\partial y}(\bar{t}, \varphi_1(\bar{t})) \varphi_1^{(r)}(\bar{t}) \right. \right. \\ & \left. \left. - \frac{\partial h}{\partial y}(t, \varphi_2(t)) \varphi_2^{(r)}(t) + \frac{\partial h}{\partial y}(\bar{t}, \varphi_2(\bar{t})) \varphi_2^{(r)}(\bar{t}) \right| \cdot \right. \\ & \left. (\gamma(|t - \bar{t}|))^{-1} \right\} \leq \\ & \leq f \left(\sum_{k=0}^n \left| \varphi_1^{(k)}(a) - \varphi_2^{(k)}(a) \right| + \sup_{t \neq \bar{t}} \frac{\left| \varphi_1^{(n)}(t) - \varphi_1^{(n)}(\bar{t}) - \varphi_2^{(n)}(t) + \varphi_2^{(n)}(\bar{t}) \right|}{\gamma(|t - \bar{t}|)} \right), \end{aligned}$$

for all $t, \bar{t} \in [a, b]$.

Omitting the sign supremum on the left hand side of the above inequality and putting $t = x, \bar{t} = \bar{x}$, by (10), (11), we get

$$\begin{aligned} & \sum_{k=0}^r |h_k(a, z_1, 0, \dots, 0) - h_k(a, z_2, 0, \dots, 0)| + |u_r(x, z_1, 0, \dots, 0) - u_r(x, z_2, 0, \dots, 0) - \\ & u_r(\bar{x}, \varphi_1(\bar{x}), \dots, \varphi_1^{(r-1)}(\bar{x})) + u_r(\bar{x}, \varphi_2(\bar{x}), \dots, \varphi_2^{(r-1)}(\bar{x})) - \frac{\partial h}{\partial y}(\bar{x}, \varphi_1(\bar{x}))\varphi_1^{(r)}(\bar{x}) + \\ & \quad + \frac{\partial h}{\partial y}(\bar{x}, \varphi_2(\bar{x}))\varphi_2^{(r)}(\bar{x})| \cdot \\ & (\gamma(|x - \bar{x}|))^{-1} \leq f(|z_1 - z_2|), \end{aligned}$$

and, consequently,

$$\begin{aligned} & \left| u_r(x, z_1, 0, \dots, 0) - u_r(x, z_2, 0, \dots, 0) - u_r(\bar{x}, \varphi_1(\bar{x}), \dots, \varphi_1^{(r-1)}(\bar{x})) + \right. \\ & \left. u_r(\bar{x}, \varphi_2(\bar{x}), \dots, \varphi_2^{(r-1)}(\bar{x})) - \frac{\partial h}{\partial y}(\bar{x}, \varphi_1(\bar{x}))\varphi_1^{(r)}(\bar{x}) + \frac{\partial h}{\partial y}(\bar{x}, \varphi_2(\bar{x}))\varphi_2^{(r)}(\bar{x}) \right| \\ & \leq f(|z_1 - z_2|)\gamma(|x - \bar{x}|). \end{aligned}$$

Hence, letting $\bar{x} \rightarrow x$ in so obtained inequality, by the right-hand side continuity in zero of γ and (12), we have

$$\frac{1}{(n-k+1)!} \left| \frac{\partial h}{\partial y}(x, z_1) - \frac{\partial h}{\partial y}(x, z_2) \right| \leq 0, \quad z_1, z_2 \in \mathbb{R},$$

whence,

$$\frac{\partial h}{\partial y}(x, z_1) = \frac{\partial h}{\partial y}(x, z_2), \quad z_1, z_2 \in \mathbb{R}.$$

Thus h is linear with respect to the second variable and, consequently, there exist functions $\alpha, \beta: [a, b] \rightarrow \mathbb{R}$ such that

$$h(x, y) = \alpha(x)y + \beta(x), \quad x \in [a, b], \quad y \in \mathbb{R}. \quad (13)$$

Since $h(\cdot, y) \in W_\gamma^r[a, b]$ for all $y \in \mathbb{R}$, by (13),

$$\beta(x) = h(x, 0), \quad \alpha(x) = h(x, 1) - \beta(x), \quad x \in I,$$

therefore $\alpha, \beta \in W_\gamma^r[a, b]$, which completes the proof.

In the sequel the following definition plays a crucial role:

Definition 3.1 ([10], Definition 1). Let X and Y be two metric spaces. We say that a mapping $F: X \rightarrow Y$ is *uniformly bounded* if, for any $t > 0$, there exists a real number $f(t)$ such that for any nonempty set $B \subset X$, we have

$$\text{diam}B \leq t \Rightarrow \text{diam}F(B) \leq f(t).$$

The main result of this paper reads as follows:

Theorem 3.2. Let $a, b \in \mathbb{R}$, $n, r \in \mathbb{N}$, $a < b$, $n \geq r$, be fixed and let a function $h: [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be such that for any $x \in [a, b]$ the function $h(x, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ is r -times differentiable and its r -th derivatives satisfy the Lipschitz condition on \mathbb{R} . If the composition operator h of the generator h maps the space $W_\gamma^n[a, b]$ into $W_\gamma^r[a, b]$, $n \geq r$, and is uniformly bounded, then there exist $\alpha \in W_\gamma^r[a, b]$ and $\beta \in W_\gamma^r[a, b]$ such that

$$h(x, y) = \alpha(x)y + \beta(x), \quad x \in [a, b], \quad y \in \mathbb{R}.$$

and

$$H(\varphi)(x) = \alpha(x)\varphi(x) + \beta(x), \quad \varphi \in W_\gamma^n[a, b], \quad (x \in [a, b]).$$

Proof. Take any $t \geq 0$ and arbitrary $\varphi_1, \varphi_2 \in W_\gamma^n[a, b]$ such that

$$\|\varphi_1 - \varphi_2\|_{W_\gamma^n[a, b]} \leq t.$$

Since $\text{diam}\{\varphi_1, \varphi_2\} \leq t$, by the uniform boundedness of H , we have,

$$\text{diam}H(\{\varphi_1, \varphi_2\}) \leq f(t),$$

that is

$$\|H(\varphi_1) - H(\varphi_2)\|_{W_\gamma^r[a, b]} = \text{diam}H(\{\varphi_1, \varphi_2\}) \leq f(\|\varphi_1 - \varphi_2\|_{W_\gamma^n[a, b]})$$

and the result follows from Theorem 1.

Remark 3.1. If the function $f: [0, \infty) \rightarrow [0, \infty)$ in Definition 3.1 is right continuous at 0 and $f(0) = 0$ (or if only $f(0^+) = 0$), then, clearly, the uniform boundedness of the involved operator reduces to its uniform continuity.

Remark 3.2. Taking in Theorem 3.2 $\gamma: [0, d] \rightarrow [0, \infty)$ such that $\gamma'_+(0)$ is finite, $f(t) = ct$, for some constant $c \geq 0$, we get the main result of Knop [5].

Remark 3.3. Theorem 3.2 also extends the result of Lupa [6], where $\gamma: [0, d] \rightarrow [0, \infty)$ is such that $\gamma'_+(0) = \infty$, $f(t) = ct$ ($c \geq 0$) and $r = n$.

4. Conclusions

We show that if a Nemytskij composition operator H mapping the space $W_\gamma^n[a, b]$ of n -times differentiable functions with the n -th derivative satisfying

a generalized Hölder condition into $W_V^r[a, b]$ ($r \leq n$) is uniformly bounded or uniformly continuous, then its generator is an affine function with respect to the second variable. This significantly extends the result of [5, 6] where much stronger Lipschitz norm continuity of H is assumed.

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