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FINDING THE EXPECTED REVENUES IN MARKOV NETWORKS WITH POSITIVE AND NEGATIVE CUSTOMERS AT A STATIONARY REGIME

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Abstract. Finding the expected revenues in the queueing systems (QS) of open Markov G-networks of two types, with positive and negative customers and with positive customers and signals, has been described in the paper. A negative customer arriving to the system destroys one positive customer if at least one is available in the system, thus reducing the number of positive customers in the system by one. The signal, coming into an empty system (where there are no positive customers), does not have any impact on the network and immediately disappears from it. Otherwise, if the system is not empty, when it receives a signal, the following events can occur: the incoming signal instantly moves the positive customer from one QS into another with a certain probability, or with the other probability, the signal is triggered as a negative customer.

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1. Introduction

Queueing systems with revenues at a stationary regime were introduced into consideration in [1], but networks were introduced in [2]. The review of the results obtained by the systems and queueing networks (QN) at a stationary regime are contained in [3]. They are devoted to finding the mean revenues in the network systems that depend only on their states and do not depend on the time and solved the problem of finding the optimal service rates of customers in QS by the dynamic programming method. Revenues from transitions between network states were not considered. QN, with revenues at a non-stationary regime, have been studied in [4, 5]. In such a network, in the transition of a positive customer of QS to another, the last customer gets some generally random revenue, and the revenue of the first QS is reduced accordingly by that amount. Revenues from transitions between network states were either depending on the QN state and time, or were a random variable (RV) with specified moments of the first and second orders. In review paper [6] in the monograph [7], the results of research, optimization and selection of the optimal management strategies in Markov networks with the revenues are presented, various applications of management strategies as probability forecasting models of the expected revenues in information and telecommunication systems and networks (for instance, when service requests on the server brings revenue to service company), insurance companies, logistics transportation systems, industrial systems and other facilities are described.

As it is known, any Markov QN functioning can be described by the Markov chain with continuous time and with a larger or countable number of states. The simplest case of Markov chain with a small number of states and constant revenues from the transitions between states has been considered in the monograph [8].

In recent years, attention has been paid to the study of Markov chains with revenues and various features: with bounded waiting time of claims and unreliable QS [9-11]. Time-dependent expressions for the expected revenues have been obtained. However, this was done with one important limitation: QS should have been able to operate in a heavy-traffic regime, i.e. at any time in the QS it should serve at least one customer. The current paper covers a Markov network with revenues, positive and negative customers, as well as with signals, in the case when revenues from transitions between network states are random variables (RV) with given mean values; this restriction was removed. Expressions for finding expected revenues in the network systems in a unit time at a stationary regime have been derived.

It should be noted that Markov QN with positive and negative customers (without revenues) were introduced by E. Gelenbe [12] as models of behavior of computer viruses in the information and telecommunication systems and networks and are called G-networks. In the same paper, the condition of existence of a stationary regime is presented. The paper shows that, as for almost all Markov chains, the stationary distribution of probabilities has a multiplicative form.

Markov networks with positive customers and signals were introduced in [13] and investigated at a stationary regime. The action of signals consists in immediately moving a positive customer with a specified probability of the system into some other network system.

The signal can work as a trigger, which does not destroy the customer, but only moves it immediately with a given probability from one system to another network system as a negative customer. Signals (triggers) are used to control the load in the network.

Consider an open queueing G-network with n single-line QSs. An incoming flow of positive (regular) customers with rate (intensity) λ_{0i}^{+} and Poisson flow of negative customers with rate λ_{0i}^{-}, $i=\overline{1,n}$ are coming into QS S_i from the outside (from QS S_0). All flows of customers entering the network are independent. The service time of the positive customers in the QS S_i is exponentially distributed with the rate μ_i, $i=\overline{1,n}$. A negative customer, arriving to some QS in which there is at least one positive customer, instantly destroys (deletes, removes from the network) one of customers. On the assumption of an exponential distribution of service time of positive customers, we would not care about what exact customer is destroyed [13]. After that, positive customer immediately leaves out the network without any servicing by the QS. Thus, each QS of the network can serve only regular customers.

The positive customer, serviced by S_i at moment time t, with the probability $p_{ij}^{+}(t)$ is sent to QS S_j as a positive customer, with probability $p_{ij}^{-}(t)$ - as a negative customer, and with probability $p_{i0}(t)=1-\sum_{j=1}^{n}\left(p_{ij}^{+}(t)+p_{ij}^{-}(t)\right)$ the customer leaves the network to the external environment (QS S_0), $i, j=\overline{1,n}$. By the network state we mean a vector $k(t)=(k,t)=(k_1(t),k_2(t),...,k_n(t))=(k_1,k_2,...,k_n,t)$ of dimension $n+1$, where $k_i(t)$ - the count of customers in the system S_i at time t, $i=\overline{1,n}$. Processes $k(t)$, $k_i(t)$, $i=\overline{1,n}$, are Markov chains with continuous time and a countable number of states. We will assume that Markov chain $k_i(t)$ is ergodic, having a stationary distribution, $i=\overline{1,n}$.

2. Analysis of the expected revenues in a network with positive and negative customers when the revenues from the network transitions between the states are given random variables with given mean values

Let RV ξ_i - service time of the customer by the system S_i, distributed exponentially with the distribution function $F_{\xi_i}(t)=1-e^{\mu_i t}, i=\overline{1,n}.$ Let's consider the dynamics of revenue changes of the *i*-th system. Let at the initial time the revenue of QS S_i is equal to v_{i0}; $V_i(k,t)$ - revenue of the *i*-th QS at time *t*, if at the initial time the network was at state *k*. The revenue of its QS at moment time $t+\Delta t$ can be represented as

$$V_i(k,t+\Delta t)=V_i(k,t)+\Delta V_i(k,t,\Delta t). \tag{1}$$

To find the revenues of QS S_i, specify the conditional probabilities of the events that may occur during Δt, $i=\overline{1,n}$. The following cases are possible:

1) a positive customer will come to the i-th QS from the external environment with probability $\lambda_{0i}^{+}\Delta t + o(\Delta t)$, it will bring a revenue of the amount r_{0i}, where r_{0i} - RV with expected value (E) $E\{r_{0i}\} = a_{0i}$, $i = \overline{1,n}$;
2) a negative customer will come to QS S_i from the external environment with probability $\lambda_{0i}^{-}\Delta t + o(\Delta t)$, it will bring a loss of the amount $-\bar{r}_{0i}$, $\bar{r}_{0i}$ - RV with $E\{\bar{r}_{0i}\} = \bar{a}_{0i}$, $i = \overline{1,n}$;
3) a positive customer after servicing by the i-th system will come out of the network to the external environment with probability $\mu_i p_{i0} u(k_i(t))\Delta t + o(\Delta t)$, a revenue of the i-th QS will decrease by an amount R_{i0}, where R_{i0} - RV with $E\{R_{i0}\} = b_{i0}$, $u(x) = \begin{cases} 1, x > 0 \\ 0, x \le 0 \end{cases}$ - Heaviside function, $i = \overline{1,n}$;
4) a positive customer will move from the i-th QS to the j-th QS with probability $\mu_i p_{ij}^{+} u(k_i(t))\Delta t + o(\Delta t)$, $i, j = \overline{1,n}$, $i \ne j$; a revenue of the i-th QS will decrease by an amount $R_{ij}(\xi_i)$, and revenue of the j-th QS will increase by this amount, $$E\{R_{ij}(\xi_i)\} = \int_0^\infty R_{ij}(t)dF_{\xi_i}(t) = \mu_i \int_0^\infty R_{ij}(t)e^{-\mu_i t}dt = a_{ij}, i, j = \overline{1,n}, i \ne j; \qquad i, j = \overline{1,n},$$ $i \ne j$;
5) a positive customer will move to the i-th QS with probability $\mu_j p_{ji}^{+} u(k_j(t))\Delta t + o(\Delta t)$, a revenue of the i-th QS will increase by an amount $R_{ji}(\xi_j)$, and revenue of the j-th QS will decrease by this amount, $E\{R_{ji}\} = a_{ji}$, $j = \overline{1,n}$, $j \ne i$;
6) a positive customer, after finishing the service by the i-th QS, will move to the j-th QS as a negative customer with probability $\mu_i p_{ij}^{-}\Delta t + o(\Delta t)$, $i, j = \overline{1,n}, i \ne j$; by such a transition, a revenue of the i-th QS will decrease by an amount $\bar{R}_{ij}$, $\bar{R}_{ij}$ - RV with $E\{\bar{R}_{ij}\} = \bar{c}_{ij}$, $i, j = \overline{1,n}, i \ne j$;
7) no state changes of the i-th system will happen on the time interval $[t, t + \Delta t)$ with probability $1 - \sum_{j=1}^{n}\left[\lambda_{0j}^{+} + \lambda_{0j}^{-} + \mu_j\right]\mathrm{u}(\mathrm{k_j}(t))\Delta t + o(\Delta t)$, $j = \overline{1,n}$, $j \ne i$;
8) the i-th system at each small time interval Δt will increase its revenue by the amount of $r_i\Delta t$ (due to servicing some customers), r_i - RV with $E\{r_i\} = d_i$, $i = \overline{1,n}$.

From the aforesaid follows

$$\Delta V_i(k,t,\Delta t)=\begin{cases} r_{0i}+r_i\Delta t \text{ with prob. } \lambda_{0i}^{+}\Delta t+o(\Delta t),\\ -\overline{r}_{0i}+r_i\Delta t \text{ with prob. } \lambda_{0i}^{-}\Delta t+o(\Delta t),\\ -R_{i0}+r_i\Delta t \text{ with prob. } \mu_i p_{i0}u(k_i(t))\Delta t+o(\Delta t),\\ -R_{ij}(\xi_i)+r_i\Delta t \text{ with prob. } \mu_i p_{ij}^{+}u(k_i(t))\Delta t+o(\Delta t),\ j=\overline{1,n},\ j\neq i,\\ R_{ji}(\xi_j)+r_i\Delta t \text{ with prob. } \mu_j p_{ji}^{+}u(k_j(t))\Delta t+o(\Delta t),\ j=\overline{1,n},\ j\neq i,\\ -\overline{R}_{ji}+r_i\Delta t \text{ with prob. } \mu_i p_{ij}^{-}u(k_j(t))\Delta t+o(\Delta t),\ j=\overline{1,n},\ j\neq i,\\ r_i\Delta t \text{ with prob. } 1-\sum_{j=1}^{n}\left[\lambda_{0j}^{+}+\lambda_{0j}^{-}+\mu_j\right]u(k_j(t))\Delta t+o(\Delta t). \end{cases} \quad (2)$$

We find the expression for the change of expected revenue in *i*-th QS at time *t*. According to (2), the following can be written for the expected value:

$$E\{\Delta V_i(k,t,\Delta t)\}=(a_{0i}+d_i\Delta t)(\lambda_{0i}^{+}\Delta t+o(\Delta t))+(-\overline{a}_{0i}+d_i\Delta t)(\lambda_{0i}^{-}\Delta t+o(\Delta t))+$$
$$+(-b_{i0}+d_i\Delta t)(\mu_i p_{i0}u(k_i(t))\Delta t+o(\Delta t))+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[(-a_{ij}+d_i\Delta t)(\mu_j p_{ji}^{+}u(k_i(t))\Delta t+o(\Delta t))\right]+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[(a_{ji}+d_i\Delta t)(\mu_j p_{ji}^{+}u(k_j(t))\Delta t+o(\Delta t))\right]+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[(-\overline{c}_{ij}+d_i\Delta t)(\mu_i p_{ij}^{-}u(k_j(t))\Delta t+o(\Delta t))\right]+$$
$$+d_i\Delta t\left(1-\sum_{j=1}^{n}\left[\lambda_{0j}^{+}+\lambda_{0j}^{-}+\mu_j\right]u(k_j(t))\Delta t+o(\Delta t)\right)=$$
$$=\left[d_i+a_{0i}\lambda_{0i}^{+}-\overline{a}_{0i}\lambda_{0i}^{-}-b_{i0}\mu_i p_{i0}u(k_i(t))\right]\Delta t+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[-a_{ij}\mu_i p_{ij}^{+}u(k_j(t))+a_{ji}\mu_j p_{ji}^{+}u(k_j(t))-\overline{c}_{ij}\mu_i p_{ij}^{-}u(k_j(t))\right]\Delta t+o(\Delta t),\ i=\overline{1,n}. \quad (3)$$

Therefore, as it follows from (1), (3),

$$E\{V_i(k,t+\Delta t)\}=v_i(k,t+\Delta t)=E\{V_i(k,t)\}=v_i(k,t)+E\{\Delta V_i(k,t,\Delta t)\}, \quad (4)$$

Integrating both sides of (4) from 0 to *t*:

$$v_i(k,t)=v_i(k,0)+(\lambda_{0i}^{+}a_{0i}-\lambda_{0i}^{-}\overline{a}_{0i}+d_i)t-\mu_i\left(b_{i0}p_{i0}+\sum_{\substack{j=1\\ j\neq i}}^{n}a_{ij}p_{ij}^{+}\right)\int_0^t u(k_i(\tau))d\tau+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[a_{ji}\mu_j p_{ij}^{+}-\overline{c}_{ij}\mu_i p_{ij}^{-}\right]\int_0^t u(k_i(\tau))d\tau,\ i=\overline{1,n}. \quad (5)$$

Obviously, $\int_0^t u(k_i(\tau))d\tau$ - total time during which on the interval of time $[0,t]$ customers count in the i-th QS is more than zero. Because the $k_i(t)$ is an ergodic stationary random process, $\int_0^t u(k_i(\tau))d\tau$ is also ergodic process as a real measurable function of an ergodic process $k_i(t)$ [14]. Therefore

$$\frac{1}{t}\int_0^t u(k_i(\tau))d\tau \xrightarrow[t\to\infty]{a.s.} Mu(k_i) = 1\cdot P(k_i>0)+0\cdot P(k_i=0) = P(k_i>0) =$$
$$= 1-P(k_i=0) = q_i, \tag{6}$$

where k_i - customers count in the i-th QS at a stationary regime.

Value $\bar{v}_i = \lim\limits_{t\to\infty}\frac{v_i(k,t)}{t}$ characterizes the expected revenue of the i-th QS at a unit of time at a stationary regime. Then, dividing both sides of (5) by t, tends t to infinity, and because of (6) we shall have:

$$\bar{v}_i = \lambda_{0i}^{+}a_{0i} - \lambda_{0i}^{-}\bar{a}_{0i} + d_i - \mu_i q_i\left(b_{i0}p_{i0} + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}p_{ij}^{+}\right) +$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n} q_j\left[a_{ji}\mu_i p_{ij}^{+} - \bar{c}_{ij}\mu_i p_{ij}^{-}\right], i=\overline{1,n}. \tag{7}$$

The E. Gelenbe paper [12] shows that values $q_i, i=\overline{1,n}$, are found by solving the following system of nonlinear equations:

$$q_i = \frac{\lambda_{0i}^{+} + \sum\limits_{j=1}^{n}\mu_j p_{ji}^{+} q_j}{\lambda_{0i}^{-} + \sum\limits_{j=1}^{n}\mu_j p_{ji}^{+} q_j + \mu_i}, i=\overline{1,n}. \tag{8}$$

For comparison, we give the relation for the value of revenue (7) of the i-th QS in the case when in the network there are no negative customers [6, 7], which we denote by v_i:

$$v_i = \lambda a_{0i}\tilde{p}_{0i} + d_i - \mu_i b_{i0}\tilde{p}_{i0} - \mu_i\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}\tilde{p}_{ij} + \sum_{\substack{j=1\\ j\neq i}}^{n}\mu_j a_{ji}p_{ij}, i=\overline{1,n},$$

where: λ - the intensity of the incoming simplest flow of customers (arriving rate) to the network; $\tilde{p}_{0i}$ - arrive probability to the i-th QS, $i=\overline{1,n}$, $\sum_{i=1}^{n}\tilde{p}_{0i}=1$; $\tilde{p}_{ij}$ -

probability that a positive customer, served by S_i, moves to QS S_j as a positive customer, $\sum_{j=1}^{n} \widetilde{p}_{ij} = 1, i = \overline{1,n}$; $\widetilde{p}_{i0}$ - probability that, a customer leaves the network to the external environment, $\widetilde{p}_{i0} = 1 - \sum_{j=1}^{n} \widetilde{p}_{ij}, i = \overline{1,n}.$

Let's find the difference $v_i - \bar{v}_i$. Note that $p_{ij}^+ = \widetilde{p}_{ij}$ and $\lambda_{0i}^+ = \lambda \widetilde{p}_{0i}$ $i, j = \overline{1,n}$. Taking into account the expressions for p_{0i} we obtain

$$v_i - \bar{v}_i = \lambda a_{0i} \widetilde{p}_{0i} + d_i - \mu_i b_{i0} \widetilde{p}_{i0} - \mu_i \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \widetilde{p}_{ij} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \mu_j a_{ji} p_{ij} - \lambda_{0i}^+ a_{0i} + \lambda_{0i}^- \bar{a}_{0i} - d_i +$$

$$+ \mu_i q_i \left(b_{i0} p_{i0} + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} p_{ij}^+ \right) - \sum_{\substack{j=1 \\ j \neq i}}^{n} q_j \left[a_{ji} \mu_i p_{ij}^+ - \bar{c}_{ij} \mu_i p_{ij}^- \right] =$$

$$= \lambda_{0i}^- \bar{a}_{0i} - \mu_i b_{i0} \left(q_i p_{i0} + p_{i0} \right) +$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} \left\{ \mu_j a_{ji} p_{ij} - \mu_i \left[\left(1 - q_i\right) a_{ij} p_{ij}^+ + q_j \left(a_{ji} p_{ij}^+ - \bar{c}_{ij} p_{ij}^- \right) \right] \right\}.$$

i.e. presence of the negative customers reduces the revenues of QS S_i by this amount.

3. Analysis of the expected revenues in a network with positive customers and signals

Consider now a G-network with positive customers and signals [13, 15]. Network description is given in [2, 3]. The signal, coming in an empty system S_i (in which there are no positive customers), does not have any impact on the network and immediately disappears from it. Otherwise, if the system S_i is not empty, when it receives a signal, the following events can occur: incoming signal instantly moves the positive customer from the system S_i into the system S_j with probability q_{ij}, in this case, the signal is referred to as a trigger; or with probability $q_{i0} = 1 - \sum_{j=1}^{n} q_{ij}$ the signal is triggered by a negative customer and destroys in QS S_i the positive customer. The state of the network defined by the vector $k(t) = (k, t) = (k_1, k_2, ..., k_n, t)$, where k_i - the customers count at the moment of time t at the system S_i, $i = \overline{1,n}$.

To find the revenues of the *i*-th QS, we write the conditional probabilities of the events that may occur during Δt, $i=\overline{1,n}$. The following cases are possible:

– a signal will arrive to the *i*-th QS from the external environment with probability $\lambda_{0i}^{-}q_{i0}\Delta t+o(\Delta t)$; it will trigger as a negative customer and will destroy a positive customer in QS S_i, that will bring a loss in an amount $-\bar{r}_{0i}$, $\bar{r}_{0i}$ - RV with $E\{\bar{r}_{0i}\}=\bar{a}_{0i}$, $i=\overline{1,n}$;

– signal arriving to QS S_i will immediately move a positive customer from system S_i to system S_j with probability $\lambda_{0i}^{-}q_{ij}u(k_i(t))\Delta t+o(\Delta t)$, $i,j=\overline{1,n}$, $i\neq j$; by such a transition a revenue of the *i*-th QS will decrease by an amount $\bar{r}_{0i}$, revenue of the *j*-th QS will increase by this amount, where $\bar{r}_{0i}$ - RV with $E\{\tilde{r}_{0i}\}=\tilde{a}_{0i}$, $i=\overline{1,n}$;

– a customer from system S_i will move to QS S_j as signal, if there were no customers, with probability $\mu_i(1-u(k_i(t)))p_{ij}^{-}\Delta t+o(\Delta t)$, by such a transition a revenue of the *i*-th QS will decrease by an amount $\overline{R}_{ij}$, $\overline{R}_{ij}$ - RV with $E\{\overline{R}_{ij}\}=\bar{a}_{ij}$, $i=\overline{1,n}$;

– a positive customer after finishing the service in QS S_i will move to QS S_j as a signal, which is triggered as negative customer, will destroy in QS S_j a positive customer; the probability of this event is $\mu_i p_{ij}^{-}q_{j0}\Delta t+o(\Delta t)$; by such a transition a revenue of S_j will decrease by an amount $\widetilde{R}_{ji}(\xi_j)$, and a revenue of S_i will decrease by this amount, $E\{\widetilde{R}_{ji}(\xi_j)\}=\bar{c}_{ji}$, $j=\overline{1,n}$, $j\neq i$;

– a positive customer after finishing the service by QS S_i will move to QS S_j as a signal, which will immediately move a positive customer from system S_j to system S_s; the probability of this event is $\mu_i p_{ij}^{-}q_{js}u(k_j(t))\Delta t+o(\Delta t)$, $i=\overline{1,n}$, $j=\overline{1,n}, i\neq j$; by such a transition a revenue of the *i*-th QS and system S_j will decrease by an amount R_{ijs}, and for the system S_s will increase by this amount, where R_{ijs} – RV with $E\{R_{ijs}\}=c_{ijs}$, $i,j,s=\overline{1,n}$, $i\neq j$, $s\neq i$;

– the network state will not change on the interval Δt with probability $1-\left(\lambda_{0j}^{+}+\lambda_{0j}^{-}+\mu_j\right)u(k_j(t))\Delta t+o(\Delta t)$.

The revenue changes of S_i on the interval $[t, t+\Delta t]$ may be written as

$$\Delta V_i(k,t,\Delta t)=\begin{cases} r_{0i}+r_i\Delta t \text{ with prob. } \lambda_{0i}^{+}\Delta t+o(\Delta t),\\ -\overline{r}_{0i}+r_i\Delta t \text{ with prob. } \lambda_{0i}^{-}q_{i0}\Delta t+o(\Delta t),\\ -\hat{r}_{0i}+r_i\Delta t \text{ with prob. } \lambda_{0i}^{-}q_{ij}u(k_i(t))\ \Delta t+o(\Delta t),\\ -R_{i0}+r_i\Delta t \text{ with prob. } \mu_i p_{i0}u(k_i(t))\ \Delta t+o(\Delta t),\\ -\overline{R}_{ij}+r_i\Delta t \text{ with prob. } \mu_i\left(1-u(k_i(t))\right)p_{ij}^{-}\Delta t+o(\Delta t), i,j=\overline{1,n}, j\neq i,\\ -R_{ij}(\xi_i)+r_i\Delta t \text{ with prob. } \mu_i p_{ij}^{+}u(k_i(t))\Delta t+o(\Delta t), i,j=\overline{1,n}, j\neq i,\\ R_{ji}(\xi_j)+r_i\Delta t \text{ with prob. } \mu_j p_{ji}^{+}u(k_j(t))\Delta t+o(\Delta t),\ i,j=\overline{1,n}, j\neq i,\\ -\tilde{R}_{ji}(\xi_j)+r_i\Delta t \text{ with prob. } \mu_i p_{ij}^{-}q_{j0}\Delta t+o(\Delta t),\ i,j=\overline{1,n}, j\neq i,\\ -R_{ijs}+r_i\Delta t \text{ with prob. } \mu_i p_{ij}^{-}q_{js}u(k_j(t))\Delta t+o(\Delta t), i,s=\overline{1,n}, j\neq i, s\neq i,\\ r_i\Delta t \text{ with prob. } 1-\left[\lambda_{0j}^{+}+\lambda_{0j}^{-}+\mu_j\right]\ u(k_j(t))\Delta t+o(\Delta t). \end{cases} \quad (9)$$

Taking into account (9), for the expected value of the revenue changes can be written:

$$\begin{aligned} E\left\{\Delta V_i^{(s)}(k,t,\Delta t)\right\} &= (a_{0i}+d_i\Delta t)\ \left(\lambda_{0i}^{+}\Delta t+o(\Delta t)\right)+\\ &+\left(-\overline{a}_{0i}+d_i\Delta t\right)\ \left(\lambda_{0i}^{-}q_{i0}\Delta t+o(\Delta t)\right)+\\ &+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\left(-\tilde{a}_{0i}+d_i\Delta t\right)\ \left(\lambda_{0i}^{-}q_{ij}u(k_i(t))\Delta t+o(\Delta t)\right)\right]+\\ &+\left(-b_{i0}+d_i\Delta t\right)\ \left(\mu_i p_{i0}u(k_i(t))\Delta t+o(\Delta t)\right)+\\ &+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\left(-\overline{a}_{ij}+d_i\Delta t\right)\left(\mu_i(1-u(k_i(t)))p_{ij}^{-}\Delta t+o(\Delta t)\right)\right]+\\ &+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\left(-a_{ij}+d_i\Delta t\right)\ \left(\mu_i p_{ij}^{+}u(k_i(t))\Delta t+o(\Delta t)\right)\right]+\\ &+\sum_{\substack{j=1\\ j\neq i}}^{n}\left[\left(a_{ji}+d_i\Delta t\right)\ \left(\mu_j p_{ji}^{+}u(k_j(t))\Delta t+o(\Delta t)\right)\right]+ \end{aligned}$$

$$+\sum_{j=1}^{n}\Big[\big(-\overline{a}_{ji}+d_i\Delta t\big)\ \big(\mu_i p_{ij}^{-}q_{j0}\Delta t+o(\Delta t)\big)\Big]+$$

$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{s=1\\ s\neq i}}^{n}\Big[\big(c_{ijs}+d_i\Delta t\big)\ \big(\mu_i p_{ij}^{-}u\big(k_j(t)\big)q_{js}\Delta t+o(\Delta t)\big)\Big]+$$

$$+d_i\Delta t\left(1-\sum_{j=1}^{n}\Big[\lambda_{0j}^{+}+\big(\lambda_{0j}^{-}+\mu_j\big)\Big]u\big(k_j(t)\big)\ \Delta t+o(\Delta t)\right)=$$

$$=\big(a_{0i}\lambda_{0i}^{+}-\overline{a}_{0i}\lambda_{0i}^{-}q_{i0}-b_{i0}\mu_i p_{i0}u\big(k_i(t)\big)+d_i\big)\Delta t+$$

$$+\sum_{j=1}^{n}\Big[-\tilde{a}_{0i}\lambda_{0i}^{-}q_{ij}u\big(k_i(t)\big)-a_{ij}\mu_i p_{ij}^{+}u\big(k_i(t)\big)+a_{ji}\mu_j p_{ji}^{+}u\big(k_j(t)\big)-$$

$$\left.-\overline{c}_{ji}\mu_i p_{ij}^{-}q_{j0}-\sum_{\substack{s=1\\ s\neq i}}^{n}c_{ijs}\mu_i p_{ij}^{-}q_{js}u\big(k_j(t)\big)\right]\Delta t+o(\Delta t),\ i=\overline{1,n}.$$

Then, similarly as in (4), for the expected revenue of the system S_i with positive customers and signals, we obtain

$$\left(v_i^{(s)}(k,t)\right)'=E\left\{V_i^{(s)}(t)\right\}=v_i^{(s)}(k,0)+\Bigg(a_{0i}\lambda_{0i}^{+}-\overline{a}_{0i}\lambda_{0i}^{-}q_{i0}-b_{i0}\mu_i p_{i0}+d_i-$$

$$-\tilde{a}_{0i}\lambda_{0i}^{-}u\big(k_i(t)\big)\sum_{\substack{j=1\\ j\neq i}}^{n}q_{ij}-\mu_i u\big(k_i(t)\big)\sum_{\substack{j=1\\ j\neq i}}^{n}a_{ij}p_{ij}^{+}+\sum_{\substack{j=1\\ j\neq i}}^{n}a_{ji}\mu_j p_{ji}^{+}u\big(k_j(t)\big)-$$

$$\left.-\mu_i\sum_{\substack{j=1\\ j\neq i}}^{n}\overline{c}_{ji}p_{ij}^{-}q_{j0}+\mu_i\sum_{\substack{j=1\\ j\neq i}}^{n}p_{ij}^{-}u\big(k_j(t)\big)\sum_{s=1}^{n}c_{ijs}q_{js}\right)\Delta t=$$

$$=v_i^{(s)}(k,0)+\Bigg\{\lambda_{0i}^{+}a_{0i}-\lambda_{0i}^{-}\overline{a}_{0i}q_{i0}-\mu_i b_{i0}p_{i0}+d_i-\mu_i\sum_{\substack{j=1\\ j\neq i}}^{n}\overline{c}_{ji}p_{ij}^{-}q_{j0}-\Bigg(\tilde{a}_{0i}\lambda_{0i}^{-}\sum_{\substack{j=1\\ j\neq i}}^{n}q_{ij}+$$

$$\left.-\mu_i\sum_{\substack{j=1\\ j\neq i}}^{n}a_{ij}p_{ij}^{+}\right)u\big(k_i(t)\big)+\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\mu_j a_{ij}p_{ij}^{+}+\overline{c}_{ji}p_{ij}^{-}q_{j0}-\sum_{\substack{s=1\\ s\neq i}}^{n}c_{ijs}p_{ij}^{-}q_{js}\right)u\big(k_j(t)\big)\Bigg\}t,\quad (10)$$

Having integrated both sides (10) from 0 to t we obtain:

$$v_i^{(s)}(k,t) = v_i^{(s)}(k,0) + \left(\lambda_{0i}^{+} a_{0i} - \lambda_{0i}^{-} \overline{a}_{0i} q_{i0} - \mu_i b_{i0} p_{i0} + d_i - \mu_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{c}_{ji} p_{ij}^{-} q_{j0} \right) t -$$

$$- \left(\tilde{a}_{0i} \lambda_{0i}^{-} \sum_{\substack{j=1 \\ j \neq i}}^{n} q_{ij} + \mu_i \sum_{\substack{j=1 \\ j \neq i}}^{n} p_{ij} p_{ji}^{+} \right) \int_0^t u(k_i(\tau)) d\tau +$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\mu_j a_{ji} p_{ji}^{+} - \sum_{\substack{s=1 \\ s \neq i}}^{n} c_{ijs} p_{ij}^{-} q_{js} \right) \int_0^t u(k_j(\tau)) d\tau.$$

Dividing both sides of expression by t and using the relation $\frac{1}{t}\int_0^t u(k_i(\tau)) d\tau \xrightarrow[t\to\infty]{a.s.} q_i^{(s)}$, where $q_i^{(s)}$, similarly as in the preparation of equality (6), but only for the network with positive customers and signals - this is stationary probability that the i-th QS is not empty, can obtain relation for the expected revenue of the i-th QS of the considered network per unit time at a stationary regime:

$$v_i^{(s)}(k,t) = \lambda_{0i}^{+} a_{0i} - \lambda_{0i}^{-} \overline{a}_{0i} q_{i0} - \mu_i b_{i0} p_{i0} + d_i -$$

$$-\mu_i \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{c}_{ji} p_{ij}^{-} q_{j0} - \left(\tilde{a}_{0i} \lambda_{0i}^{-} \sum_{\substack{j=1 \\ j \neq i}}^{n} q_{ij} + \mu_i \sum_{\substack{j=1 \\ j \neq i}}^{n} p_{ij} p_{ji}^{+} \right) q_i^{(s)} +$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\mu_j a_{ji} p_{ij}^{+} - \sum_{\substack{s=1 \\ s \neq i}}^{n} c_{ijs} p_{ij}^{-} q_{js} \right) q_j^{(s)}, i = \overline{1,n}.$$

Finding the stationary probabilities $q_i^{(s)}$, $i = \overline{1,n}$ has been described in [13, 15].

Similarly, as previously, differences $v_i - \overline{v}_i^{(s)}$ could be found, which show how much the existence of signals reduces network revenues.

4. Conclusions

A Markov queueing network with revenues, positive and negative customers, and also signals, in the case when revenues from transitions between network states are RV with given mean values, has been carried out. Relations for finding the expected revenues in the network systems per unit time at a stationary regime have been derived.

Further investigations may be related to finding network systems revenue variances, and also clarifying for what revenue functions, expected revenues will take similar values. It is also possible to generalize obtained results for the case when arrival rates of positive and negative customers, service rates of positive customers, depend on network states and time.

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