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CALCULATING STEADY-STATE PROBABILITIES OF SINGLE-CHANNEL CLOSED QUEUEING SYSTEMS USING HYPEREXPONENTIAL APPROXIMATION

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Abstract. In this paper we propose a method for calculating steady-state probability distributions of the single-channel closed queueing systems with arbitrary distributions of customer generation times and service times. The approach based on the use of fictitious phases and hyperexponential approximations with parameters of the paradoxical and complex type by the method of moments. We defined conditions for the variation coefficients of the gamma distributions and Weibull distributions, for which the best accuracy of calculating the steady-state probabilities is achieved in comparison with the results of simulation modeling.

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1. Introduction

The purpose of this work is the analysis of a model of a single-channel closed queueing system that is employed, in particular, in the theory of communication networks and integral queueing networks [1-4]. A closed system is also known as a system with a finite number of sources or the Engset system. We assume that customers from m identical sources are fed to a queueing system. Each source can generate only one customer, and the next customer is not sent if the previous customer is not processed. The time interval from the moment at which the customer is returned to the source to the moment of the arrival of this customer to the system is the customer generation time. Intensity of the input flow of customers of a closed system depends on the number of customers in the system $\xi(t)$ at moment t and is proportional to the value $m-\xi(t)$.

Works [5-8] show that the use of hyperexponential approximation (denoted by H_l) makes it possible to determine with high accuracy the steady-state probabilities of non-Markovian queuing systems. These probabilities are determined using solutions of a system of linear algebraic equations obtained by the method of fictitious phases. To find parameters of the H_l-approximation of a certain distribution, it is sufficient to solve the system of equations of the moments method. For the values $V<1$ of the variation coefficient, roots of this system are complex-valued or paradoxical (i.e., negative or with probabilities that exceed the boundaries of the interval $[0, 1]$) but in most cases as a result of summation of probabilities of microstates, their complex-valued and paradoxical parts are annihilated.

The hyperexponential approximation method in the work [8] was used to calculate steady-state probabilities of a closed system with exponentially distributions of customer generation times. The purpose of the paper is to use the hyperexponential approximation method for calculating steady-state probabilities of single-channel closed queueing systems with arbitrary distributions of customer generation times and service times. We indicate ways to evaluate the accuracy of approach the obtained steady-state distribution to the true distribution without the need of using simulation models.

2. Equations for steady-state probabilities of the $H_4/H_l/1/m$ closed system

The hyperexponential distribution of order l is a phase-type distribution and provides for choosing one of l alternative phases by a random process. With probability y_i, the process is at the i-th phase and is in it during an exponentially distributed time with a parameter θ_i.

Suppose that the customer generation time is distributed according to the hyperexponential law H_4 with probabilities α_i and parameters $\lambda_i \, (1 \le i \le 4)$, and the service times of each customer are independent random variables distributed according to the hyperexponential law $H_l \, (l \ge 2)$ with probabilities β_s and parameters $\mu_s \, (1 \le s \le l)$. The closed system under consideration is denoted by $H_4/H_l/1/m$ and will be used for an approximate calculation of the steady-state probabilities of the $G/G/1/m$ closed system.

Let us enumerate the $H_4/H_l/1/m$ system's states as follows: $x_{0(i,j,u,v)}$ corresponds to the empty system, and i, j, u, v are the number of customers for which the generation time is in the first, second, third and fourth phase, respectively; $x_{k(i,j,u,v,s)}$ is the state, when there are k customers in the system $(1 \le k \le m)$, and i, j, u, v are the number of customers for which the generation time is in the first, second, third and fourth phase, respectively, and s is the phase number of service time. The proposed numbering of the states differs from that introduced in the works [6-8] and helps reduce the number of states of the $H_4/H_l/1/m$ closed system. We denote by $p_{0(i,j,u,v)}$ and $p_{k(i,j,u,v,s)}$, steady-state probabilities that the system is in the each of these states respectively. Since the process of changing the states of the system is Markovian with continuous time, for the steady-state probabilities $p_{0(i,j,u,v)}$ and $p_{k(i,j,u,v,s)}$ we obtain a system of linear algebraic equations that follows from the Kolmogorov system of differential equations. Here we only provide equations that correspond to states $x_{0(i,j,u,v)}$ and a normalization condition:

$$-m\lambda_1 p_{0(m,0,0,0)} + \alpha_1 \sum_{i=1}^{l} \mu_i p_{1(m-1,0,0,0,i)} = 0, -m\lambda_2 p_{0(0,m,0,0)} + \alpha_2 \sum_{i=1}^{l} \mu_i p_{1(0,m-1,0,0,i)} = 0,$$

$$-m\lambda_3 p_{0(0,0,m,0)} + \alpha_3 \sum_{i=1}^{l} \mu_i p_{1(0,0,m-1,0,i)} = 0, -m\lambda_4 p_{0(0,0,0,m)} + \alpha_4 \sum_{i=1}^{l} \mu_i p_{1(0,0,0,m-1,i)} = 0,$$

$$-\left(i\lambda_1 + (m-i)\lambda_2\right) p_{0(i,m-i,0,0)} + \alpha_1 \sum_{j=1}^{l} \mu_j p_{1(i-1,m-i,0,0,j)} + \alpha_2 \sum_{j=1}^{l} \mu_j p_{1(i,m-i-1,0,0,j)} = 0,$$

$$-\left(i\lambda_1 + (m-i)\lambda_3\right) p_{0(i,0,m-i,0)} + \alpha_1 \sum_{j=1}^{l} \mu_j p_{1(i-1,0,m-i,0,j)} + \alpha_3 \sum_{j=1}^{l} \mu_j p_{1(i,0,m-i-1,0,j)} = 0,$$

$$-\left(i\lambda_1 + (m-i)\lambda_4\right) p_{0(i,0,0,m-i)} + \alpha_1 \sum_{j=1}^{l} \mu_j p_{1(i-1,0,0,m-i,j)} + \alpha_4 \sum_{j=1}^{l} \mu_j p_{1(i,0,0,m-i-1,j)} = 0,$$

$$-\left(i\lambda_2 + (m-i)\lambda_3\right) p_{0(0,i,m-i,0)} + \alpha_2 \sum_{j=1}^{l} \mu_j p_{1(0,i-1,m-i,0,j)} + \alpha_3 \sum_{j=1}^{l} \mu_j p_{1(0,i,m-i-1,0,j)} = 0,$$

$$-\left(i\lambda_2 + (m-i)\lambda_4\right) p_{0(0,i,0,m-i)} + \alpha_2 \sum_{j=1}^{l} \mu_j p_{1(0,i-1,0,m-i,j)} + \alpha_4 \sum_{j=1}^{l} \mu_j p_{1(0,i,0,m-i-1,j)} = 0,$$

$$-\left(i\lambda_3 + (m-i)\lambda_4\right) p_{0(0,0,i,m-i)} + \alpha_3 \sum_{j=1}^{l} \mu_j p_{1(0,0,i-1,m-i,j)} + \alpha_4 \sum_{j=1}^{l} \mu_j p_{1(0,0,i,m-i-1,j)} = 0,$$

$$1 \le i \le m-1;$$

$$-\left(i\lambda_1 + j\lambda_2 + (m-i-j)\lambda_3\right) p_{0(i,j,m-i-j,0)} + \alpha_1 \sum_{s=1}^{l} \mu_s p_{1(i-1,j,m-i-j,0,s)} +$$

$$+ \alpha_2 \sum_{s=1}^{l} \mu_s p_{1(i,j-1,m-i-j,0,s)} + \alpha_3 \sum_{s=1}^{l} \mu_s p_{1(i,j,m-i-j-1,0,s)} = 0,$$

$$-\left(i\lambda_1 + j\lambda_2 + (m-i-j)\lambda_4\right) p_{0(i,j,0,m-i-j)} + \alpha_1 \sum_{s=1}^{l} \mu_s p_{1(i-1,j,0,m-i-j,s)} +$$

$$+ \alpha_2 \sum_{s=1}^{l} \mu_s p_{1(i,j-1,0,m-i-j,s)} + \alpha_4 \sum_{s=1}^{l} \mu_s p_{1(i,j,0,m-i-j-1,s)} = 0,$$

$$-\left(i\lambda_1 + j\lambda_3 + (m-i-j)\lambda_4\right) p_{0(i,0,j,m-i-j)} + \alpha_1 \sum_{s=1}^{l} \mu_s p_{1(i-1,0,j,m-i-j,s)} +$$

$$+ \alpha_3 \sum_{s=1}^{l} \mu_s p_{1(i,0,j-1,m-i-j,s)} + \alpha_4 \sum_{s=1}^{l} \mu_s p_{1(i,0,j,m-i-j-1,s)} = 0,$$

$$-\left(i\lambda_2 + j\lambda_3 + (m-i-j)\lambda_4\right) p_{0(0,i,j,m-i-j)} + \alpha_2 \sum_{s=1}^{l} \mu_s p_{1(0,i-1,j,m-i-j,s)} +$$

$$+ \alpha_3 \sum_{s=1}^{l} \mu_s p_{1(0,i,j-1,m-i-j,s)} + \alpha_4 \sum_{s=1}^{l} \mu_s p_{1(0,i,j,m-i-j-1,s)} = 0,$$

$$1 \le i \le m-2, \quad 1 \le j \le m-i-1;$$

$$-\left(i\lambda_1 + j\lambda_2 + s\lambda_3 + (m-i-j-s)\lambda_4\right) p_{0(i,j,s,m-i-j-s)} +$$

$$+\alpha_1 \sum_{u=1}^{l} \mu_u p_{1(i-1,j,s,m-i-j-s,u)} + \alpha_2 \sum_{u=1}^{l} \mu_u p_{1(i,j-1,j,s,m-i-j-s,u)} +$$

$$+\alpha_3 \sum_{u=1}^{l} \mu_u p_{1(i,j,s-1,m-i-j-s,u)} + \alpha_4 \sum_{u=1}^{l} \mu_u p_{1(i,j,s,m-i-j-s-1,u)} = 0,$$

$$1 \le i \le m-3, \; 1 \le j \le m-i-2, \; 1 \le s \le m-i-j-1;$$

$$\ldots$$

$$\sum_{i=0}^{m} \sum_{j=0}^{m-i} \sum_{s=0}^{m-i-j} p_{0(i,j,s,m-i-j-s)} + \sum_{k=1}^{m} \sum_{i=0}^{m-k} \sum_{j=0}^{m-k-i} \sum_{s=0}^{m-k-i-j} \sum_{u=1}^{l} p_{k(i,j,s,m-k-i-j-s,u)} = 1. \qquad (1)$$

In addition to the equations given in (1), we write the equations corresponding to the following states separately:

$x_{1(m-1,0,0,0,i)}$, $x_{1(0,m-1,0,0,i)}$, $x_{1(0,0,m-1,0,i)}$, $x_{1(0,0,0,m-1,i)}$, $1 \le i \le l$;

$x_{k(m-k,0,0,0,i)}$, $x_{k(0,m-k,0,0,i)}$, $x_{k(0,0,m-k,0,i)}$, $x_{k(0,0,0,m-k,i)}$, $2 \le k \le m-1$, $1 \le i \le l$;

$x_{1(i,m-1-i,0,0,j)}$, $x_{1(i,0,m-1-i,0,j)}$, $x_{1(i,0,0,m-1-i,j)}$, $x_{1(0,i,m-1-i,0,j)}$,

$x_{1(0,i,0,m-1-i,j)}$, $x_{1(0,0,i,m-1-i,j)}$, $1 \le i \le m-2$, $1 \le j \le l$;

$x_{k(i,m-k-i,0,0,j)}$, $x_{k(i,0,m-k-i,0,j)}$, $x_{k(i,0,0,m-k-i,j)}$, $x_{k(0,i,m-k-i,0,j)}$,

$x_{k(0,i,0,m-k-i,j)}$, $x_{k(0,0,i,m-k-i,j)}$, $2 \le k \le m-2$, $1 \le i \le m-k-1$, $1 \le j \le l$;

$x_{1(i,j,m-1-i-j,0,s)}$, $x_{1(i,j,0,m-1-i-j,s)}$, $x_{1(i,0,j,m-1-i-j,s)}$,

$x_{1(0,i,j,m-1-i-j,s)}$, $1 \le i \le m-3$, $1 \le j \le m-i-2$, $1 \le s \le l$;

$x_{k(i,j,m-k-i-j,0,s)}$, $x_{k(i,j,0,m-k-i-j,s)}$, $x_{k(i,0,j,m-k-i-j,s)}$,
$x_{k(0,i,j,m-k-i-j,s)}$, $2 \le k \le m-3,\ 1 \le i \le m-k-2,\ 1 \le j \le m-k-i-1,\ 1 \le s \le l;$
$x_{1(i,j,s,m-1-i-j-s,u)}$, $1 \le i \le m-4,\ 1 \le j \le m-i-3,\ 1 \le s \le m-i-j-2,\ 1 \le u \le l;$
$x_{k(i,j,s,m-k-i-j-s,u)}$, $2 \le k \le m-4,\ 1 \le i \le m-k-3,\ 1 \le j \le m-k-i-2,$
$1 \le s \le m-k-i-j-1,\ 1 \le u \le l;$ $x_{m(0,0,0,0,i)}$, $1 \le i \le l.$

Solving the system of linear algebraic equations, we find the steady-state probabilities p_k of the presence in the closed queueing system of k customers using the formulas

$$p_0 = \sum_{i=0}^{m}\sum_{j=0}^{m-i}\sum_{s=0}^{m-i-j} p_{0(i,j,s,m-i-j-s)},$$
$$p_k = \sum_{i=0}^{m-k}\sum_{j=0}^{m-k-i}\sum_{s=0}^{m-k-i-j}\sum_{u=1}^{l} p_{k(i,j,s,m-k-i-j-s,u)}, \quad 1 \le k \le m. \tag{2}$$

3. Numerical results

The method of potentials was used in [9] to construct an algorithm that makes it possible to determine the steady-state distribution of the number of customers for a single-channel closed queueing system with exponentially distributed customer generation times and an arbitrary distribution of service times. This method is not suitable for the *G*/*G*/1/m closed queueing systems.

For the *M*/*G*/1/m closed queueing systems the deviation $\Delta_{sim} = \sum_{k=0}^{m} | p_k - p_{k(sim)} |$ of distribution $\{p_{k(sim)}\}$, obtained using the GPSS World simulation system [10], from the distribution $\{p_k\}$, obtained using the method of potentials, exceeds 10^{-4}.

In this section, we consider the *G*/*G*/1/m closed queueing systems with the gamma distributions and Weibull distributions and determine the values of the variation coefficients of these distributions, for which the condition $\Delta_{(6,5)} < 10^{-4}$ holds when calculating the steady-state distribution of the number of customers in the system. If this condition is fulfilled, the distribution $\{p_{k(6)}\}$ is a more accurate approximation to the true steady-state distribution $\{p_k\}$ than the distribution obtained using the GPSS World simulation system. Here $\Delta_{(6,5)} = \sum_{k=0}^{m} | p_{k(6)} - p_{k(5)} |$ gives an opportunity to estimate the deviation of distributions $\{p_{k(6)}\}$ from distributions $\{p_{k(5)}\}$, and $p_{k(l)}$ are values of probabilities p_k obtained using the $H_4/H_l/1/m$ system as an approximation of the *G*/*G*/1/m system.

Let $\Gamma(V)$ and $W(V)$ denote the gamma distribution and Weibull distribution with coefficients of variation V. For convenience, we introduce the following numbering of closed queueing systems: assign the $\Gamma(V)/\Gamma(V_2)/1/7$, $\Gamma(V)/W(V_2)/1/7$, $W(V)/W(V_2)/1/7$, $W(V)/\Gamma(V_2)/1/7$, $\Gamma(V_1)/\Gamma(V)/1/7$, $W(V_1)/\Gamma(V)/1/7$, $W(V_1)/W(V)/1/7$, $\Gamma(V_1)/W(V)/1/7$ systems numbers 1-8 respectively. We take $m=7$, $E(T_\lambda)=1$, $E(T_{sv})=0.3$. Here $E(T_\lambda)$ and $E(T_{sv})$ denote the mean of the customer generation times and the service times, respectively.

The numerical results are presented in Table 1. Of all the intervals corresponding to the $\Gamma(V)$ distributions, the values $V=0.5$ and $V=1/\sqrt{2}$ of the variation coefficient should be excluded, since for the $\Gamma(0.5)$ distribution it is not possible to construct approximations with the help of hyperexponential distributions of order higher than the third, and hyperexponential approximations do not exist for the $\Gamma(1/\sqrt{2})$ distribution.

Table 1. List of the $G/G/1/7$ closed systems with the gamma distributions and Weibull distributions for which condition $\Delta_{(6,5)}<10^{-4}$ holds

V	System number	Values of V_2 for systems with numbers 1-4	System number	Values of V_1 for systems with numbers 5-8
0.001	1	[0.51, 1.3]	5	[0.8, 100]
	2	[0.5, 1.1]	6	[0.4, 10]
	3	[0.5, 1.1]	7	[0.4, 10]
	4	[0.51, 1.3]	8	[0.8, 100]
0.1	1	[0.51, 1.3]	5	[0.8, 100]
	2	[0.5, 1.1]	6	[0.4, 10]
	3	[0.5, 1.1]	7	[0.4, 10]
	4	[0.51, 1.3]	8	[0.8, 100]
0.2	1	[0.4, 1.3]	5	0.4, [0.8, 100]
	2	[0.4, 1.1]	6	[0.3, 10]
	3	[0.4, 1.1]	7	[0.3, 10]
	4	0.4, [0.51, 1.3]	8	0.4, [0.8, 100]
0.3	1	[0.3, 1.4]	5	[0.3, 0.4], [0.8, 100]
	2	[0.3, 1.2]	6	[0.3, 10]
	3	[0.2, 1.2]	7	[0.3, 10]
	4	[0.2, 0.4], [0.51, 1.4]	8	[0.3, 0.4], [0.8, 100]
0.4	1	[0.2, 1.4]	5	[0.2, 0.4], [0.8, 100]
	2	[0.2, 1.2]	6	[0.2, 10]
	3	[0.001, 1.2]	7	[0.2, 10]
	4	[0.001, 0.4], [0.51, 1.4]	8	[0.2, 0.4], [0.8, 100]
0.5	3	[0.001, 1.2]	7	[0.001, 10]
	4	[0.001, 0.4], [0.51, 1.4]	8	–
0.6	1	[0.99, 1.3]	5	[0.001, 0.4], [0.8, 100]
	2	[1.1, 1.2]	6	[0.001, 10]
	3	[0.001, 1.2]	7	[0.001, 10]
	4	[0.001, 0.4], [0.51, 1.5]	8	[0.001, 0.4], [0.8, 100]

Cont. Table 1

V	System number	Values of V_2 for systems with numbers 1-4	System number	Values of V_1 for systems with numbers 5-8
0.7	1	[0.8, 1.4]	5	[0.001, 0.4], [0.8, 100]
	2	[0.99, 1.2]	6	[0.001, 10]
	3	[0.001, 1.3]	7	[0.001, 10]
	4	[0.001, 0.4], [0.51, 1.5]	8	[0.001, 0.4], [0.8, 100]
0.8	1	[0.001, 1.5]	5	[0.001, 0.4], [0.7, 100]
	2	[0.001, 1.3]	6	[0.001, 10]
	3	[0.001, 1.3]	7	[0.001, 10]
	4	[0.001, 0.4], [0.51, 1.5]	8	[0.001, 0.4], [0.8, 100]
0.9	1	[0.001, 1.5]	5	[0.001, 0.4], [0.7, 100]
	2	[0.001, 1.3]	6	[0.001, 10]
	3	[0.001, 1.3]	7	[0.001, 10]
	4	[0.001, 0.4], [0.51, 1.5]	8	[0.001, 0.4], [0.8, 100]
1.1	1	[0.001, 1.5]	5	[0.001, 0.4], [0.6, 100]
	2	[0.001, 1.3]	6	[0.001, 10]
	3	[0.001, 1.3]	7	[0.001, 10]
	4	[0.001, 0.4], [0.51, 1.5]	8	[0.001, 0.4], [0.6, 100]
1.2	1	[0.001, 1.5]	5	[0.001, 0.4], [0.6, 100]
	2	[0.001, 1.3]	6	[0.001, 10]
	3	[0.001, 1.3]	7	[0.3, 10]
	4	[0.001, 0.4], [0.51, 1.5]	8	[0.3, 0.4], [0.6, 100]
1.3	1	[0.001, 1.5]	5	[0.001, 0.4], [0.6, 100]
	2	[0.001, 1.3]	6	[0.001, 10]
	3	[0.001, 1.3]	7	[0.7, 10]
	4	[0.001, 0.4], [0.51, 1.5]	8	[0.8, 100]
1.4	1	[0.001, 1.5]	5	[0.3, 0.4], [0.7, 100]
	2	[0.001, 1.3]	6	[0.3, 10]
	3	[0.001, 1.3]	7	–
	4	[0.001, 0.4], [0.51, 1.5]	8	–
1.5	1	[0.001, 1.5]	5	[0.8, 2.3]
	2	[0.001, 1.3]	6	[0.6, 10]
	3	[0.001, 1.3]	7	–
	4	[0.001, 0.4], [0.51, 1.5]	8	–
1.6	1	[0.001, 1.5]	5	–
	2	[0.001, 1.3]	6	–
	3	[0.001, 1.3]	7	–
	4	[0.001, 0.4], [0.51, 1.5]	8	–
10	1	[0.001, 1.4]	5	–
	2	[0.001, 1.3]	6	–
	3	[0.001, 1.3]	7	–
	4	[0.001, 0.4], [0.51, 1.5]	8	–

4. Conclusions

This paper shows that the application of hyperexponential approximation of distributions of customer generation times, and the service times allow us to calculate steady-state probabilities of the single-channel closed queueing systems with arbitrary distributions of customer generation times and service times, with high accuracy (higher than in the case of using simulation models). We find these probabilities using solutions of a system of linear algebraic equations obtained by the method of fictitious phases. To obtain parameters of H_l-approximation of a certain distribution, it is necessary to solve the system of equations of the moments method.

Computing deviations $\Delta_{(6,5)}$ allows us to track the accuracy of approaching distributions $\{p_{k(l)}\}$ to the true distribution $\{p_k\}$ without the need of using simulation models. We defined conditions for the variation coefficients of the gamma distributions and Weibull distributions, for which the best accuracy of calculateing the steady-state probabilities is achieved compared with the case of using simulation models.

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