

THE FINITE DIFFERENCE METHOD ON ADAPTIVE MESH FOR SINGULARLY PERTURBED NONLINEAR 1D REACTION DIFFUSION BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study singularly perturbed nonlinear reaction-diffusion equations. The asymptotic behavior of the solution is examined. The difference scheme which is accomplished by the method of integral identities with using of interpolation quadrature rules with weight functions and remainder term integral form is established on adaptive mesh. Uniform convergence and stability of the difference method are discussed in the discrete maximum norm. The discrete scheme shows that orders of convergent rates are close to 2. An algorithm is presented, and some problems are solved to validate the theoretical results.

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Keywords: boundary value problem, singularly perturbed problem, finite difference method

1. Introduction

This study is concerned with following the singularly perturbed boundary value problem in which a second-order derivative is multiplied by a small positive parameter ε^2

$$-\varepsilon^2 u'' + a(x)u(x) = f(x, u); \quad 0 \leq x \leq l, \quad (1)$$

$$u(0) = \kappa_0, \quad u(l) = \kappa_1, \quad (2)$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, κ_0 and κ_1 are given constants. $a(x)$ and $f(x, u)$ ($(x, u) \in [0, l] \times \mathbb{R}$) are given sufficiently smooth functions. Moreover

$$\frac{\partial f}{\partial u} \geq p > 0 \text{ in } [0, l] \times \mathbb{R}.$$

Reaction-diffusion equations are typical mathematical models in many areas of science and engineering. Nuclear reactivity, population dynamics, biological

systems, a control theory, fluid dynamics, chemical processes and finance are among these [1-4]. For example, travelling waves of chemical concentration were dealt with by fractional reaction-diffusion equations [5]. Furthermore, the following equation has been presented for models of the spread of invasive species

$$\frac{\partial u(x, t)}{\partial t} = D \frac{\partial^2 u(x, t)}{\partial^2 t} + f(u(x, t)),$$

where $u(x, t)$ is the population density and $x \in \mathbb{R}$ [4]. Another sample has been introduced for blow-up profiles

$$\begin{aligned} u_t &= \Delta u + e^u, \quad x \in \Omega, \quad t > 0, \\ u(x, 0) &= u_0(x) \geq 0, \end{aligned}$$

where $|x| < R$ and u_0 is non-negative [6].

Different forms of reaction-diffusion equations have been located in literature. Some authors have taken interest in numerical solutions of these equations. Therefore, there are many different types of numerical approaches that have been presented. The implicit Runge-Kutta method was used for time-dependent reaction-diffusion equations [7]. The explicit Euler discretized is considered for nonlinear form [3]. The implicit Galerkin-Legendre spectral method is applied to the two dimensional fractional form [2]. We see the singularly perturbed form of these equations in recent studies. Standard and stabilised finite difference methods are carried out on a Shishkin mesh for semilinear singularly perturbed reaction-diffusion boundary value problems [8]. By using the method of two-steps expansions, nonlocal equations are studied [9]. Anisotropic nonconforming finite element methods are considered with the interpolation postprocessing technique [10]. Considering the Richardson technique, an improved scheme is constructed [11]. The Richardson extrapolation method with the fourth-order stable central difference method were carried out for the self-adjoint type of these equations [12]. Using the compact finite difference method, error bounds are obtained for 1D forms [13]. The standard implicit Euler method and the HODIE compact fourth order finite difference scheme were used [14]. Sobolev gradient methods were used with Dirichlet boundary conditions [15]. By taking into account the finite element method, sign-changing solutions are obtained [16]. The Hybrid finite difference scheme and Numerov scheme are established on a Shishkin type mesh [17]. By carrying out boundary layer analysis, numerical solutions are obtained in a quasi-uniform mesh [18]. The finite difference scheme is constructed for one and two dimensional types [19]. By using asymptotic behavior of the solution, a domain decomposition method is analyzed into three overlapping subdomains [20]. Besides, for delay form of these equations, a fitted finite difference scheme is constructed by using the Numerov's method [1] and a difference scheme is established on a piecewise equidistant mesh [21].

Singularly perturbed problems have boundary layers in which the solution changes quickly. In such problems, when $\varepsilon \rightarrow 0$, the boundary layer is getting thinner. Algorithms which are established by traditional numerical techniques are unstable. That's why we need the reliable numerical methods.

Our goal in this paper is to present the robust numerical method for singularly perturbed reaction-diffusion equations on adaptive mesh.

The draft of this research is as follows: In section 2, the properties of the solution for (1)-(2) are handled. In section 3, the difference scheme is constructed on adaptive mesh. In section 4, error approximations are obtained with respect to Bakhvalov mesh transition points. In section 5, the theoretical results are tested on some numerical examples.

2. Continuous problem

In this section, we give the asymptotic behavior of the solution and its derivatives, which is required in analysis of the numerical method. For the (1)-(2) nonlinear problem, using the Taylor expansion, we can write

$$f(x, u) = f(x, 0) + \frac{\partial f(x, \tilde{u})}{\partial u} u,$$

where $\tilde{u} = \gamma u$, $0 < \gamma < 1$. Also we get $F(x) = f(x, 0)$ and $b(x) = \frac{\partial f(x, \tilde{u})}{\partial u}$. Then

$$-\varepsilon^2 u'' + [a(x) - b(x)]u = F(x).$$

If we get $A(x) = a(x) - b(x)$, we obtain

$$-\varepsilon^2 u'' + A(x)u = F(x), \tag{3}$$

where

$$A(x) = a(x) - \frac{\partial f(x, \tilde{u})}{\partial u} \geq \alpha > 0, \tag{4}$$

$$F(x) = f(x, 0) \geq 0.$$

We give the following lemmas for the properties of the solution.

Lemma 1. [22] Let $v(x)$ be a function and satisfy the following conditions:

$$Lv = F(x) \geq 0,$$

$\kappa_0 \geq 0, \kappa_1 \geq 0$. Then $v(x) \geq 0$.

Lemma 2. [22] For a $v(x) \in C[0, l] \cap C^2(0, l)$ function, the following estimate is held:

$$|v(x)| \leq |v(0)| + |v(l)| + \alpha^{-1} \max_{0 \leq s \leq l} |Lv(s)|, \quad 0 \leq x \leq l, \tag{5}$$

Lemma 3. For the solution of the (1)-(2) nonlinear problem, the following estimations are true:

$$|u(x)| \leq C, \quad 0 < x < l; \quad A(x), F(x) \in C[0, l], \quad (6)$$

$$|u'(x)| \leq C \left\{ 1 + \frac{1}{\varepsilon} \left(e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\varepsilon}} \right) \right\}, \quad 0 < x < l, \quad A(x), F(x) \in C[0, l], \quad (7)$$

Proof. The proof of the lemma is done in a similar manner [22].

3. Discretization

In this section, we proceed to construct the difference scheme. Let $\bar{\omega}_N$ be any non-uniform mesh on $[0, l]$.

$$\bar{\omega}_N = \{0 = x_1 < x_2 < \dots < x_{N-1} < x_N = l\},$$

and

$$\bar{\omega}_N = \omega_N \cup \{x_1 = 0, x_N = l\}.$$

Before constructing the difference scheme, we define some notation for the mesh functions. For any mesh function we defined on $\bar{\omega}_N$:

$$v_i = v(x_i), \quad v_{x,i} = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\hat{x},i} = \frac{v_i - v_{i-1}}{h_i}, \quad h_i = x_i - x_{i-1},$$

$v_{\bar{x}x,i} = \frac{1}{\bar{h}_i} (v_{x,i} - v_{\hat{x},i})$ with $\bar{h}_i = \frac{1}{2}(h_i + h_{i+1})$. Node points x_i are specified as \leftarrow

$$x_i = \begin{cases} -\alpha^{-1}\varepsilon \ln \left(1 - (1 - \varepsilon) \frac{4i}{N} \right), & i = 0, 1, \dots, \frac{N}{4}, \quad x_i \in [0, \sigma_1], \quad \sigma_1 < \frac{l}{4}; \\ -\alpha^{-1}\varepsilon \ln \left(1 - \left(1 - e^{-\frac{\alpha l}{4\varepsilon}} \right) \frac{4i}{N} \right), & i = 0, 1, \dots, \frac{N}{4}, \quad x_i \in [0, \sigma_1], \quad \sigma_1 = \frac{l}{4}; \\ \sigma_1 + \left(i - \frac{N}{4} \right) h^{(1)}, & i = \frac{N}{4} + 1, \dots, \frac{3N}{4}, \quad x_i \in [\sigma_1, \sigma_2], \quad h^{(1)} = \frac{2(\sigma_2 - \sigma_1)}{N}; \\ \sigma_2 - \alpha^{-1}\varepsilon \ln \left(1 - (1 - \varepsilon) \frac{4 \left(i - \frac{3N}{4} \right)}{N} \right), & i = \frac{3N}{4} + 1, \dots, N, \quad x_i \in [\sigma_2, l]; \\ \sigma_2 - \alpha^{-1}\varepsilon \ln \left(1 - \left(1 - e^{-\frac{\alpha l}{4\varepsilon}} \right) \frac{4 \left(i - \frac{3N}{4} \right)}{N} \right), & i = \frac{3N}{4} + 1, \dots, N, \quad x_i \in [\sigma_2, l], \quad \sigma_2 = \frac{3l}{4}, \end{cases}$$

To generate the difference method, we begin following integral identity [23]

$$\bar{h}_i^{-1} \int_{x_{i-1}}^{x_{i+1}} (-\varepsilon^2 u'' + a(x)u(x)) \varphi_i dx = \bar{h}_i^{-1} \int_{x_{i-1}}^{x_{i+1}} f(x, u) \varphi_i dx, \quad (8)$$

where

$$\varphi_i = \begin{cases} \varphi_i^{(1)} = \frac{(x - x_{i-1})}{h_i}, & x \in (x_{i-1}, x_i), \\ \varphi_i^{(2)} = \frac{(x_{i+1} - x)}{h_{i+1}}, & x \in (x_i, x_{i+1}), \end{cases}$$

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and

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i dx = \hbar_i^{-1} \left(\frac{h_i}{2} + \frac{h_{i+1}}{2} \right) = 1.$$

By using interpolating quadrature rules in [24] for each term of (8), we obtain the following difference problem

$$-\varepsilon^2 u_{\hat{x}x,i} + a_i u_i = f(x_i, u_i) + R_i, \quad i = 1, \dots, N - 1, \quad (9)$$

$$u_0 = \kappa_0, \quad u_N = \kappa_1, \quad (10)$$

where

$$R_i = R_i^{(1)} + R_i^{(2)}, \quad (11)$$

$$R_i^{(1)} = \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \varphi_i \int_{x_{i-1}}^{x_{i+1}} \frac{d^2 u(\xi)}{d\xi^2} T_0(x - \xi) d\xi,$$

$$R_i^{(2)} = -\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)] u(x) \varphi_i dx$$

$$+ \hbar_i^{-1} \left\{ \int_{x_{i-1}}^{x_{i+1}} [f(x, u) - f(x_i, u)] \varphi_i dx + \int_{x_{i-1}}^{x_{i+1}} [f(x_i, u) - f(x_i, u_i)] \varphi_i dx \right\}.$$

Hence, we can write the difference problem for the approximate solution y

$$-\varepsilon^2 y_{\hat{x}x,i} + a_i y_i = f(x_i, y_i), \quad i = 1, \dots, N - 1, \quad (12)$$

$$y_0 = y_N = 0. \quad (13)$$

4. Error estimates

To investigate the uniform convergence of the difference method, let u_i be the solution (9)-(10) and y_i be the solution of (12)-(13). The approximate error $z_i = y_i - u_i$, $x_i \in \omega_N$ is the solution of following problem

$$\varepsilon^2 z_{\hat{x}x,i} - a_i z_i = f(x_i, y_i) - f(x_i, u_i), \quad i = 1, \dots, N - 1,$$

$$z_0 = z_N = 0.$$

By modifying $f(x_i, y_i) - f(x_i, u_i)$, we can write

$$lz_i = -\varepsilon^2 z_{\hat{x}x,i} + A_i z_i = R_i, \quad i = 1, N-1, \quad z_0 = z_N = 0, \quad (14)$$

where A_i is denoted in (3) and R_i is given by (11).

Theorem 1. The solution of the difference problem (12)-(13) is uniformly convergent to the solution of the problem (1)-(2) with respect to ε in $\bar{\omega}_N$ under condition $A(x) \in C^1[0, l]$. The following estimate is held

$$\|y - u\|_{C(\bar{\omega}_N)} \leq Ch_i.$$

Proof. For the difference operator lv_i , if $lv_i \geq 0$ ($i = 1, 2, \dots, N-1$), $v_0 \geq 0$, $v_N \geq 0$, we can show $v_i \geq 0$, $i = 0, 1, \dots, N$ from the maximum principle. If Lemma 2 is applied to the problem (14), the following estimation can be written

$$\|z\|_{C(\bar{\omega}_N)} \leq \alpha^{-1} \|R\|_{C(\bar{\omega}_N)}. \quad (15)$$

For R_i , under the condition $A(x) \in C^1[0, l]$

$$|R_i| \leq Ch_i, \quad i = 1, 2, \dots, N-1. \quad (16)$$

Hence,

$$\begin{aligned} |R_i| \leq & a_i \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \varphi_i \int_{x_{i-1}}^{x_{i+1}} \left| \frac{du(\xi)}{dx} \right| T_0(x - \xi) d\xi \\ & + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} |a(x) - a(x_i)| \cdot |u(x)| \varphi_i dx \\ & + \hbar_i^{-1} \left\{ \int_{x_{i-1}}^{x_{i+1}} |f(x, u) - f(x_i, u)| \varphi_i dx + \int_{x_{i-1}}^{x_{i+1}} |f(x_i, u) - f(x_i, u(x_i))| \varphi_i dx \right\}. \end{aligned}$$

To show (16), it is enough to show that the convergence rate of the following expression is $O(\hbar_i)$

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} |f(x_i, u) - f(x_i, u(x_i))| \varphi_i dx.$$

By using the mean value theorem, the condition $\left| \frac{\partial f(x_i, \tilde{u})}{\partial u} \right| \leq C$ and

$$|u'(x)| \leq C \left\{ 1 + \frac{1}{\varepsilon} \left(e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\varepsilon}} \right) \right\},$$

we can write

$$\begin{aligned} \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} |f(x_i, u) - f(x_i, u(x_i))| \varphi_i dx &= \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial f(x_i, \tilde{u})}{\partial u} \right| |u(x) - u(x_i)| \varphi_i dx, \\ &\leq C \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} dx \varphi_i \int_{x_{i-1}}^{x_{i+1}} \left| \frac{du(\xi)}{dx} \right| d\xi, \\ &\leq C \int_{x_{i-1}}^{x_i} \left\{ 1 + \frac{1}{\varepsilon} \left(e^{-\frac{\sqrt{\alpha}\xi}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-\xi)}{\varepsilon}} \right) \right\} d\xi, \\ &\leq C \left\{ h_i + (\sqrt{\alpha})^{-1} \left[e^{-\frac{\sqrt{\alpha}x_{i-1}}{\varepsilon}} - e^{-\frac{\sqrt{\alpha}x_i}{\varepsilon}} - e^{-\frac{\sqrt{\alpha}(l-x_i)}{\varepsilon}} + e^{-\frac{\sqrt{\alpha}(l-x_{i-1})}{\varepsilon}} \right] \right\}. \end{aligned}$$

We estimate separately R_i in $[0, \sigma_1]$, $[\sigma_1, \sigma_2]$ and $[\sigma_2, l]$. First, in $[0, \sigma_1]$, for $\sigma_1 < \frac{1}{4}$

$$\begin{aligned} h_i = x_i - x_{i-1} &= \alpha^{-1} \varepsilon \left[-\ln \left(1 - \left(1 - e^{-\frac{\alpha l}{4\varepsilon}} \right) \frac{4i}{N} \right) + \ln \left(1 - \left(1 - e^{-\frac{\alpha l}{4\varepsilon}} \right) \frac{4(i-1)}{N} \right) \right], \\ &\leq 4\alpha^{-1} (1 - \varepsilon) N^{-1}. \end{aligned}$$

Besides $e^{-\frac{\sqrt{\alpha}x_{i-1}}{\varepsilon}} - e^{-\frac{\sqrt{\alpha}x_i}{\varepsilon}} \leq 4\alpha^{-1} (1 - \varepsilon) N^{-1}$ is obtained. Similarly, $e^{-\frac{\sqrt{\alpha}(l-x_{i-1})}{\varepsilon}} - e^{-\frac{\sqrt{\alpha}(l-x_i)}{\varepsilon}} \leq 4\alpha^{-1} (1 - \varepsilon) N^{-1}$ is found. If $\sigma_1 = \frac{1}{4}$,

$$\begin{aligned} h_i = x_i - x_{i-1} &= \alpha^{-1} \varepsilon \left[-\ln \left(1 - \left(1 - e^{-\frac{\alpha l}{4\varepsilon}} \right) \frac{4i}{N} \right) + \ln \left(1 - \left(1 - e^{-\frac{\alpha l}{4\varepsilon}} \right) \frac{4(i-1)}{N} \right) \right], \\ &\leq \alpha^{-1} \left(1 - e^{-\frac{\alpha l}{4\varepsilon}} \right) N^{-1} \leq \alpha^{-1} \varepsilon \frac{\alpha l}{4\varepsilon} 4N^{-1} = lN^{-1}, \end{aligned}$$

is found. Thus it is proven $|R_i| \leq CN^{-1}$ in $[0, \sigma_1]$. For $[\sigma_1, \sigma_2]$, it is written

$$h_i = x_i - x_{i-1} = \frac{2(\sigma_2 - \sigma_1)}{N} = \frac{2(l - 2\sigma_1)}{N} = 2((l - 2\sigma_1)N^{-1}).$$

$h_i \leq lN^{-1}$ for $\sigma_1 < \frac{1}{4}$. $h_i = lN^{-1}$ is obtained for $\sigma_1 = \frac{1}{4}$. Therefore it is proven that $|R_i| \leq CN^{-1}$. Similarly, it is shown that $|R_i| \leq CN^{-1}$ in $[\sigma_2, l]$. The proof of the theorem is completed.

We now consider the necessary conditions that the convergence rate is $O(h^2)$. We express that with a theorem.

Theorem 2. If $a(x) \in C^2[0, l]$, $f(x, u) \in C^2(D)$ and

$$a'(0) = a'(l) = 0,$$

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An extra parenthesis
should be removed.

σ_1
 σ_2

the convergence rate of difference scheme (12)-(13) is $O(h^2)$ and we can write as follows:

$$\|y - u\|_{C(\omega_N)} \leq Ch^2.$$

Proof. The proof of the theorem is done in a similar manner [22].

5. Results and discussion

In this section, we present the numerical outcomes for the difference scheme (12)-(13). Because of nonlinear terms, we suggest the Newton-Raphson technique for solving discretization. Applying this method, we get

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$$\begin{aligned} & -\varepsilon^2 h^{-2} \theta_i (y_{i+1}^{(n)} - 2y_i^{(n)} + y_{i-1}^{(n)}) + a_i y_i^{(n)} - y_i^{(n)} \frac{\partial f(x_i, y_i^{(n-1)})}{\partial u}, \\ -\varepsilon^2 \tilde{h}_i^{-1} (y_{x,i}^{(n)} - y_{\bar{x},i}^{(n)}) & \quad \quad \quad = f(x_i, y_i^{(n-1)}) - y_i^{(n-1)} \frac{\partial f(x_i, y_i^{(n-1)})}{\partial u}. \end{aligned}$$

Then, the difference problem (9)-(10) is modified according to the following form

$$\begin{aligned} A_i y_{i-1} - C_i y_i + B_i y_{i+1} &= -F_i, \quad i = 1, \dots, N-1, \\ y_0 = y_N &= 0, \end{aligned}$$

If the elimination method is considered, we obtain [25]

$$\begin{aligned} \varepsilon^2 \tilde{h}_i^{-1} h_i^{-1} & \quad \quad \quad A_i = \varepsilon^2 h_i^{-1} h_{i-1}^{-1}, \quad B_i = \varepsilon^2 h_i^{-1} h_{i+1}^{-1}, \\ \varepsilon^2 \tilde{h}_i^{-1} h_{i+1}^{-1} & \quad \quad \quad C_i = 2\varepsilon^2 \theta_i \theta_i + a_i - \frac{\partial f(x_i, y_i^{(n-1)})}{\partial u}, \\ \varepsilon^2 \tilde{h}_i^{-1} (h_i^{-1} + h_{i+1}^{-1}) & \quad \quad \quad F_i = f(x_i, y_i^{(n-1)}) - y_i^{(n-1)} \frac{\partial f(x_i, y_i^{(n-1)})}{\partial u}, \end{aligned}$$

where the initial process is taken as $y_i^{(0)} = -2$, $i = 1, \dots, N-1$. Furthermore, the error estimates are denoted by

$$e^N = \max_i |y_i^N - u_i^{2N}|,$$

and the convergence rates are calculated as follows

$$p_u^N = \frac{\ln(e^N / e^{2N})}{\ln 2}.$$

Example 1. We consider following the singularly perturbed nonlinear reaction-diffusion equation with boundary conditions

$$-\varepsilon^2 u''(x) + x(1-x)u(x) = u + u^2; \quad x \in (0,1),$$

$$u(0) = u(1) = 0.$$

The computational results are presented in Table 1 for different values of ε and N .

Table 1. Error approximations and convergence rates for Example 1

ε		$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-2}	e^N	0.0799526742	0.014754788	0.0035099236	0.0008603976
	e^{2N}	0.0147547885	0.0034043440	0.0008603976	0.0002140310
	p	2.4379630428	2.1157345182	2.0283641948	2.0071840245
10^{-4}	e^N	0.0828403312	0.0151691846	0.0035920895	0.0008799596
	e^{2N}	0.0151691846	0.0034750245	0.0008799596	0.0002188600
	p	2.4491897825	2.1260484965	2.0293141627	2.0074289004
10^{-6}	e^N	0.0828699525	0.0151733736	0.0035929183	0.0008801565
	e^{2N}	0.0151733736	0.0034757402	0.0008801565	0.0002189086
	p	2.4493072074	2.1261497204	2.0293240973	2.0074314626
10^{-8}	e^N	0.0828702685	0.0151734155	0.0035929266	0.0008801585
	e^{2N}	0.0151734155	0.0034757474	0.0008801585	0.0002189091
	p	2.4493087244	2.1261507342	2.0293241968	2.0074314878
10^{-10}	e^N	0.0828706110	0.0151735008	0.0035929284	0.0008801633
	e^{2N}	0.0151735008	0.0034757482	0.0008801633	0.0002189097
	p	2.4493065792	2.1261585109	2.0293170202	2.0074355070
10^{-12}	e^N	0.0828702899	0.0151734160	0.0035940285	0.0008804717
	e^{2N}	0.0151734160	0.0034765455	0.0008804717	0.0002189558
	p	2.4493090554	2.1258195359	2.0292532872	2.0076370035

Also, it is observed that the numerical solution of Example 1 has two boundary layers from Figure 1.

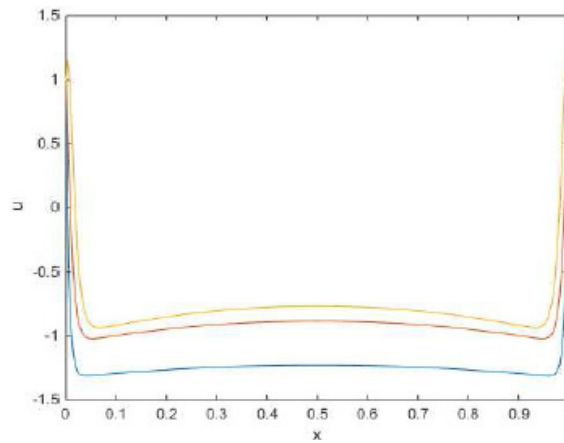


Fig. 1. Numerical solution for $\varepsilon = 10^{-4}$ and $N = 64$

Example 2. We take into account another singularly perturbed boundary value problem

$$-\varepsilon^2 u''(x) + e^x u(x) = \sin u + u; \quad x \in (0,1),$$

$$u(0) = u(1) = 0.$$

The numerical outcomes are demonstrated in Table 2.

Table 2. Error approximations and convergence rates for Example 2

ε		$N = 32$	$N = 64$	$N = 128$	$N = 256$
10^{-2}	e^N	0.0560907495	0.0131986907	0.0032395318	0.0002013206
	e^{2N}	0.0131986907	0.0032395318	0.0008061984	0.0002013206
	p	2.0873680382	2.0265376068	2.0065784439	2.0016403685
10^{-4}	e^N	0.0567253715	0.0133976816	0.0032893099	0.0008230667
	e^{2N}	0.0133976816	0.0032893099	0.0008186327	0.0002055001
	p	2.0820107768	2.0261265381	2.0064966745	2.0018704650
10^{-6}	e^N	0.0567533914	0.0134026241	0.0032904792	0.0008234119
	e^{2N}	0.0134026241	0.0032904792	0.0008189231	0.0002055862
	p	2.0821911117	2.0261458699	2.0064978773	2.0018706756
10^{-8}	e^N	0.0567571501	0.0134027487	0.0032904927	0.0008234155
	e^{2N}	0.0134027487	0.0032904927	0.0008189261	0.0002055871
	p	2.0822732420	2.0261533604	2.0064984282	2.0018707102
10^{-10}	e^N	0.0567592200	0.0134027915	0.0032904936	0.0008234155
	e^{2N}	0.0134027915	0.0032904936	0.0008189262	0.0002055871
	p	2.0823212534	2.0261575862	2.0064987317	2.0018707282
10^{-12}	e^N	0.0567605773	0.0134028187	0.0032904941	0.0008234155
	e^{2N}	0.0134028187	0.0032904941	0.0008189262	0.0002055871
	p	2.0823528207	2.0261603120	2.0064989242	2.0018707401

Figure 2 shows behavior of the numerical solution for Example 2.

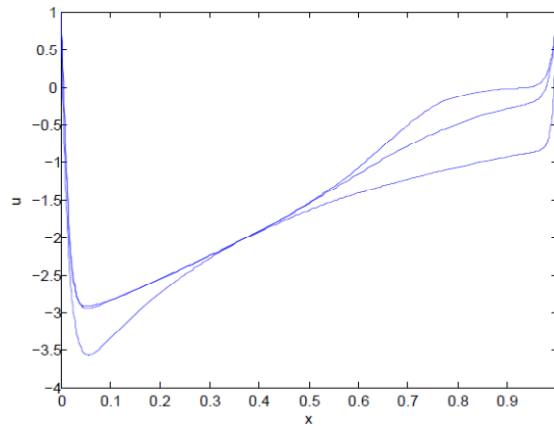


Fig. 2. Numerical solution for $\varepsilon = 10^{-6}$ and $N = 256$

The exact solutions of the Examples 1-2 are unknown. To obtain the experimental rates of convergence, we compare the computed solution with the solution on a mesh that is twice as fine. The proposed scheme is useful and simple to solve such problems. Moreover, it is uniform and yields more accurate results than some existing ones presented in the literature. Furthermore, the presented scheme can be implemented by a combination with other techniques for higher dimensional forms of these problems.

6. Conclusions

In this article, the classical difference scheme was constructed on adaptive mesh for nonlinear reaction-diffusion equations with a perturbation parameter. By using asymptotic evaluations, the convergence analysis of the method was successfully investigated. In the algorithm, the Newton-Raphson method was used because of nonlinear terms. The theoretical outcomes were tested on samples. The computational results were illustrated in tables. Based on the numerical experiments, we observe that the orders of convergence are close to 2. Numerical research can be carried out for different types such as delay, parabolic, elliptic, higher dimensional, etc.

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