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MEANS IN MONEY EXCHANGE OPERATIONS

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Abstract. It is observed that in some money exchange operations, every n-variable mean M applied by two market analysts who are acting in different countries should be self reciprocally-conjugate. The main result says that the only homogeneous weighted quasiarithmetic mean satisfying this condition is the weighted geometric mean. In the context of invariance of the geometric mean with respect to the arithmetic-harmonic mean-type mapping, the possibility of the occurring reciprocal-conjugacy in technical sciences is commented.

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1. Introduction

Motivated by some money exchange operations, we consider the n-variable means M acting in the interval $(0,\infty)$ and satisfying the condition

$$M(x_1,\dots,x_n)\,M\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right)=1,\quad x_1,\dots,x_n>0. \tag{1}$$

The role of this condition, on the model of the work of two currency market analysts, acting in two different countries, is explained in Section 2. In Section 3 we recall the basic notions concerning means [1]. For a homeomorphic mapping φ and a mean M on $(0,\infty)$, we define $M^{[\varphi]}$, a φ-conjugate mean, and we remark that a mean M satisfies condition (1), if and only if M is self-reciprocally conjugate, which holds true if and only if $M^{[\exp]}$ is an odd mean on $\mathbb{R}$ (Remark 3). In Section 4 we recall some basic facts on weighted quasiarithmetic means. In section 5 we determine the form of all odd weighted quasiarithmetic means in $\mathbb{R}$ (Proposition 1), and then, making use of Remark 3, we establish the form of all weighted quasiarithmetic means in $(0,\infty)$ satisfying condition (1) (Theorem 1). The homogeneity is expected and "good" property of a mean. Applying this result, we conclude that the geometric weighted mean $M=\mathscr{G}_{p_1,\dots,p_n}$,

$$\mathscr{G}_{p_1,\dots,p_n}(x_1,\dots,x_n):=x_1^{p_1}\cdot\dots\cdot x_n^{p_n},\quad x_1,\dots,x_n>0,$$

is the only homogeneous n-variable weighted quasiarithmetic mean with weights $p_1, \ldots, p_n > 0$; $p_1 + \ldots + p_n = 1$, satisfying condition (1) (Theorem 2).

Our considerations show that the geometric mean is the most proper tool in the money exchange operations.

The classical arithmetic mean has broad applications. The harmonic mean is applied, for instance, in kinematic where, in some special situations, it provides the correct averages of speed of a point, where it is also employed to determine the density of a homogeneous mass formed with components of given masses and densities, and it is also used in electricity and finance. Taking into account that the invariance of the geometric mean with respect to the arithmetic-harmonic mean-type mapping and its (unique) finite dimensional extension [2] (see also [3–12]), it can be expected that the considered simple model related to the currency market can be also used in technical sciences.

2. Motivation

To explain the problem, consider, for example the use of the GBP and USD in Great Britain and the United States. For the sake of clarity, let us assume that the bid and ask rates are the same, and no rounding is applied.

The UK analyst analyses the USD/GBP rate (denoted by x), where USD is the base currency and the GPB is the quote currency. The analyst in the United States analyses the rate of the same pair, but in a different ordering GBP/USD (denote it by y), where the GBP is the base currency and the USD is the quote currency. Then, to avoid arbitration, for any given time t, the condition $x_t \cdot y_t = 1$ must be met. Suppose further that both analysts record the rates at the same times t_1, t_2, $\ldots, t_n$. Then, in Great Britain, the analyst will observe the USD/GBP exchange rates

$$(x_{t_1}, x_{t_2}, \ldots, x_{t_n}),$$

and in the United States, the GBP/USD exchange rates will be:

$$(y_{t_1}, y_{t_2}, \ldots, y_{t_n}) = \left(\frac{1}{x_{t_1}}, \frac{1}{x_{t_2}}, \ldots, \frac{1}{x_{t_n}} \right).$$

Note that based on their observations, analysts know what the other analyst is observing. Analysts use many parameters/indicators to facilitate inference. The basic one, frequently used, is the arithmetic mean of the course values (hourly, daily, weekly, etc.). However it is easy to see that the equality

$$\frac{1}{n}\sum_{j=1}^{n} x_{t_j} = \frac{1}{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{x_{t_j}}}$$

holds if and only if $x_{t_1} = \ldots = x_{t_n}$, which means that analysts knowing their own arithmetic mean do not know what the mean of the other analyst is.

In this situation the following natural problem appears: characterize the means M which satisfy the relationship

$$M(x_1, x_2, \ldots, x_n) = \frac{1}{M\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)}, \qquad n \geq 2,\ x_i > 0,\ i = 1, 2, \ldots, n.$$

3. Some preliminaries and remarks

Let $I \subset \mathbb{R}$ be an interval, and let $n \in \mathbb{N}$, $n \geq 2$, be fixed.

Recall that a function $M : I^n \to I$ is called an *n-variable mean* in I, if

$$\min(x_1, \ldots, x_n) \leq M(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in I^n,$$

and the mean is called *strict,* if these inequalities are sharp for all nonconstant sequences $(x_1, \ldots, x_n) \in I^n$.

A mean $M : I^n \to I$ is called *symmetric*, if $M\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) = M(x_1, \ldots, x_n)$ for each permutation σ of $\{1, \ldots, n\}$.

A mean $M : (0,\infty)^n \to (0,\infty)$ is called *homogeneous* if

$$M(tx_1, \ldots, tx_n) = tM(x_1, \ldots, x_n), \quad t, x_1, \ldots, x_n > 0.$$

Remark 1 Let $J \subset \mathbb{R}$ be an interval, $\varphi : J \to I$ be a homeomorphic mapping of J onto I. If $M : I^n \to I$ is a mean, then the function $M^{[\varphi]} : J^n \to J$ defined by

$$M^{[\varphi]}(x_1, \ldots, x_n) := \varphi^{-1}\left(M\left(\varphi(x_1), \ldots, \varphi(x_n)\right)\right), \quad (x_1, \ldots, x_n) \in J^n,$$

is a mean in J. The mean $M^{[\varphi]}$ is called φ-conjugate of M.

Moreover, if $J = I$ and $M^{[\varphi]} = M$, then M is said to be φ-self conjugate. □

Taking for φ the reciprocal function, i.e. $\varphi(t) = \frac{1}{t}$ for $t > 0$, we get the following

Remark 2 A mean $M : (0,\infty)^n \to (0,\infty)$ satisfies condition (1) if and only if it is reciprocally self-conjugate. □

Let us note the following obvious

Remark 3 Let $n \in \mathbb{N}$, $n \geq 2$. A mean $M : (0,\infty)^n \to (0,\infty)$ satisfies (1) if and only if the exponentially conjugate mean $M^{[\exp]} : \mathbb{R}^n \to \mathbb{R}$,

$$M^{[\exp]}(t_1,\ldots,t_n) := \log M\left(e^{t_1},\ldots,e^{t_n}\right), \quad t_1,\ldots,t_n \in \mathbb{R}, \tag{2}$$

is odd, that is

$$M^{[\exp]}(-t_1,\ldots,-t_n) = -M^{[\exp]}(t_1,\ldots,t_n), \quad t_1,\ldots,t_n \in \mathbb{R}.$$

By the way, let us note that there is no even mean.

4. Weighted quasiarithmetic means

Recall a description of the quasiarithmetic means, one of the most important classes of means. These means are closely related to the conjugacy notion, as they are conjugate to the weighted arithmetic means.

Remark 4 (see for instance Chapter III in [13]) Let $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic function, and let $p_1,\ldots,p_n > 0$ be such that $p_1 + \ldots + p_n = 1$. Then the function $\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n} : I^n \to I$, given by

$$\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}(t_1,\ldots,t_n) := \varphi^{-1}\left(\sum_{j=1}^{n} p_j \varphi(t_j)\right),$$

is a strict mean in I, and it is called a *weighted quasiarithmetic mean;* the function φ is referred to as its *generator,* and $p_1,\ldots,p_n$ as its weights.

$\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}$ is symmetric iff $p_1 = \ldots = p_n = \dfrac{1}{n}$; and then this mean, denoted by $\mathscr{A}^{[\varphi]}$, is called *quasiarithmetic*. Moreover, if $\varphi, \psi : I \to \mathbb{R}$, then $\mathscr{A}^{[\psi]}_{p_1,\ldots,p_n} = \mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}$ if and only if $\psi = a\varphi + b$ for some real a, b, $a \neq 0$. □

Some related new classes of means are introduced in [14] (see also [6,8]).

Remark 5 ([13, p. 68]) Let $I = (0,\infty)$. The following conditions are equivalent:

(i) the mean $\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}$ is homogeneous;

(ii) for some $a, b, r \in \mathbb{R}$, $a \neq 0$,

$$\varphi(t) = \begin{cases} at^r + b & \text{if } r \neq 0 \\ a\log t + b & \text{if } r = 0 \end{cases}, \quad t \in (0,\infty);$$

(iii) for some $r \in \mathbb{R}$,

$$\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n} = \begin{cases} \left(\sum_{j=1}^{n} p_j x_j^r\right)^{1/r} & \text{if } r \neq 0 \\ \prod_{j=1}^{n} x_j^{p_j} & \text{if } r = 0 \end{cases}, \quad x_1,\ldots,x_n \in (0,\infty).$$

Applying the last result of Remark 1, we obtain the following

Proposition 1 *Let $n \in \mathbb{N}$, $n \geq 2$. Suppose that $\varphi : (0,\infty) \to \mathbb{R}$ is a continuous strictly monotonic function, and $p_1, \ldots, p_n > 0$ are such that $p_1 + \ldots + p_n = 1$. Then the following two conditions are equivalent:*
(i) the weighted quasiarithmetic mean $\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n} : (0,\infty)^n \to (0,\infty)$ satisfies (1), i.e.

$$\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}(x_1,\ldots,x_n)\,\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}\left(\frac{1}{x_1},\ldots,\frac{1}{x_n}\right) = 1, \quad x_1,\ldots,x_n > 0; \tag{3}$$

(ii) there are $a, b \in \mathbb{R}$, $a \neq 0$ such that φ satisfies the functional equation

$$\varphi\left(\frac{1}{x}\right) = a\varphi(x) + b, \qquad x \in (0,\infty).$$

PROOF Note that (1) holds if and only if

$$\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}(x_1,\ldots,x_n) = \frac{1}{\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}\left(\frac{1}{x_1},\ldots,\frac{1}{x_n}\right)}, \quad x_1,\ldots,x_n > 0,$$

that is, if and only if

$$\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n} = \mathscr{A}^{[\varphi\circ\beta]}_{p_1,\ldots,p_n},$$

where $\beta : (0,\infty) \to (0,\infty)$ denotes the reciprocal function $\beta(x) = \frac{1}{x}$. In view of Remark 4 (see also [13, p. 66]), this equality holds if and only if there are $a, b \in \mathbb{R}$, $a \neq 0$ such that $(\varphi \circ \beta)(x) = a\varphi(x) + b$ for all $x \in (0,\infty)$, that is if, and only if (ii) holds. ■

5. Odd weighted quasiarithmetic means and main results

By Remark 3, a mean M on $(0,\infty)$ satisfies condition (1) if and only if the mean $M^{[\exp]}$ is odd on $\mathbb{R}$. Stimulated by this fact, we prove

Proposition 2 *Let $n \in \mathbb{N}$, $n \geq 2$ and let the positive numbers $p_1, \ldots, p_n$ such that $p_1 + \ldots + p_n = 1$ be fixed. Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous strictly monotonic function.*

Then the following conditions are equivalent:

(i) the quasiarithmetic weighted mean $\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n} : \mathbb{R}^n \to \mathbb{R}$ is odd;

(ii) the function $\varphi - \varphi(0)$ is odd or, equivalently,

$$\varphi(-t) + \varphi(t) = 2\varphi(0), \qquad t \in \mathbb{R}.$$

PROOF To prove the implication (i)$\Longrightarrow$(ii), assume that $\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}$ is an odd function. By the definition of $\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}$, it means that

$$\varphi^{-1}\left(\sum_{j=1}^{n} p_j\varphi(-t_j)\right) = -\varphi^{-1}\left(\sum_{j=1}^{n} p_j\varphi(t_j)\right), \quad t_1,\ldots,t_n \in \mathbb{R}.$$

Choosing arbitrarily $j_0 \in \{2,\ldots,n-1\}$ and setting

$$p := p_1 + \ldots + p_{j_0},$$

$$t_1 = \ldots = t_{j_0} = s, \quad t_{j_0+1} = \ldots = t_n = t,$$

we hence get

$$\varphi^{-1}(p\varphi(-s) + (1-p)\varphi(-t)) = -\varphi^{-1}(p\varphi(s) + (1-p)\varphi(t)), \quad s,t \in \mathbb{R}.$$

For $s = \varphi^{-1}(u)$, $t = \varphi^{-1}(v)$, we have

$$\varphi^{-1}\left(p\varphi\left(-\varphi^{-1}(u)\right) + (1-p)\varphi\left(-\varphi^{-1}(v)\right)\right) = -\varphi^{-1}(pu + (1-p)v), \quad u,v \in \varphi(\mathbb{R}).$$

and, taking φ of both sides

$$p\varphi\left(-\varphi^{-1}(u)\right) + (1-p)\varphi\left(-\varphi^{-1}(v)\right) = \varphi\left(-\varphi^{-1}(pu + (1-p)v)\right), \quad u,v \in \varphi(\mathbb{R}).$$

Hence, setting

$$\psi := \varphi \circ \left(-\varphi^{-1}\right), \tag{4}$$

we get

$$\psi(pu + (1-p)v) = p\psi(u) + (1-p)\psi(v), \quad u,v \in \varphi(\mathbb{R}).$$

This equation and the Daróczy-Pales identity (see [15])

$$p\left(p\frac{u+v}{2} + (1-p)u\right) + (1-p)\left(pv + (1-p)\frac{u+v}{2}\right) = \frac{u+v}{2}, \quad x,y \in \mathbb{R}$$

easily imply that ψ satisfies the Jensen functional equation

$$\psi\left(\frac{u+v}{2}\right) = \frac{\psi(u) + \psi(v)}{2}, \quad u,v \in \varphi(\mathbb{R}).$$

Since ψ is continuous in the interval $\varphi(\mathbb{R})$, in view of Theorem 1, p. 315 in Kuczma [16], there are $a,b \in \mathbb{R}$, $a \neq 0$ such that

$$\psi(u) = au + b, \quad u \in \varphi(\mathbb{R}).$$

Hence, by (4),

$$\varphi \circ \left(-\varphi^{-1}\right)(u) = au + b, \quad u \in \varphi(\mathbb{R}),$$

whence, setting $u = \varphi(t)$, we get

$$\varphi(-t) = a\varphi(t) + b, \quad t \in \mathbb{R}. \tag{5}$$

Replacing here first t by $-t$, and then applying this equation again, we get

$$\begin{aligned} \varphi(t) &= a\varphi(-t) + b = a[a\varphi(t) + b] + b \\ &= a^2\varphi(t) + ab + b, \end{aligned}$$

whence

$$\left(1 - a^2\right)\varphi(t) = ab + b, \quad t \in \mathbb{R}. \tag{6}$$

Since the right side is constant, it follows that $a^2 = 1$. Consequently, either $a = 1$ or $a = -1$.

If $a = 1$, then by (6) the number b must be 0 and, by (5) we get

$$\varphi(-t) = \varphi(t), \quad t \in \mathbb{R},$$

that is a contradiction, as the function φ is strictly monotonic.

If $a = -1$, then b in (6) can be arbitrary and (5) becomes

$$\varphi(-t) = -\varphi(t) + b, \quad t \in \mathbb{R}.$$

Setting here $t = 0$ gives $\varphi(0) = -\varphi(0) + b$, whence $b = 2\varphi(0)$. Thus, by (5),

$$\varphi(-t) - \varphi(0) = -[\varphi(t) - \varphi(0)], \quad t \in \mathbb{R},$$

so the function $\varphi - \varphi(0)$ is odd, which proves the implication (i)$\Longrightarrow$(ii).

To prove the reversed implication, assume that $\varphi - \varphi(0)$ is odd. Then, the inverse function $\varphi^{-1}(y + \varphi(0))$ is also odd. Applying these facts in turn, by the definition of $\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}$, we have

$$\begin{aligned} \mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}(-t_1,\ldots,-t_n) &= \varphi^{-1}\left(\sum_{j=1}^{n} p_j\varphi(-t_j)\right) = \varphi^{-1}\left(\sum_{j=1}^{n} p_j[\varphi(-t_j) - \varphi(0)] + \varphi(0)\right) \\ &= \varphi^{-1}\left(-\sum_{j=1}^{n} p_j[\varphi(t_j) - \varphi(0)] + \varphi(0)\right) \\ &= -\varphi^{-1}\left(\sum_{j=1}^{n} p_j[\varphi(-t_j) - \varphi(0)] + \varphi(0)\right) \\ &= -\varphi^{-1}\left(\sum_{j=1}^{n} p_j\varphi(t_j)\right) = -\mathscr{A}^{[\varphi]}_{p_1,\ldots,p_n}(t_1,\ldots,t_n) \end{aligned}$$

for all $t_1, \ldots, t_n \in \mathbb{R}$, which completes the proof. ■

6. Main results

We begin this section with the following:

Remark 6 Let $n \in \mathbb{N}$, $n \geq 2$, and $J \subset (0, \infty)$ be a fixed interval. If $\gamma : J \to (0, \infty)$ is a continuous strictly monotonic function, and the real numbers $p_1, \ldots, p_n > 0$ are such that $p_1 + \ldots + p_n = 1$, then

(i) the function $\mathscr{M}^{[\gamma]}_{p_1,\ldots,p_n} : I^n \to I$ given by

$$\mathscr{M}^{[\gamma]}_{p_1,\ldots,p_n}\ (x_1, \ldots, x_n) := \gamma^{-1}\left(\prod_{j=1}^{n} [\gamma(x_j)]^{p_j}\right)$$

is a strict mean in I;

(ii) $\mathscr{M}^{[\gamma]}_{p_1,\ldots,p_n} = \mathscr{A}^{[\log\circ\gamma]}_{p_1,\ldots,p_n}$. □

PROOF Indeed, for arbitrary $x_1, \ldots, x_n > 0$,

$$\begin{aligned} \mathscr{M}^{[\gamma]}_{p_1,\ldots,p_n}\ (x_1, \ldots, x_n) &= \gamma^{-1}\left(\prod_{j=1}^{n} [\gamma(x_j)]^{p_j}\right) = \gamma^{-1}\left(\exp\left(\log \prod_{j=1}^{n} [\gamma(x_j)]^{p_j}\right)\right) \\ &= (\log\circ\gamma)^{-1}\left(\sum_{j=1}^{n} p_j (\log\circ\gamma)(x_j)\right) \\ &= \mathscr{A}^{[\log\circ\gamma]}_{p_1,\ldots,p_n}\ (x_1, \ldots, x_n), \end{aligned}$$

so (ii) holds true. Since $\mathscr{A}^{[\log\circ\gamma]}_{p_1,\ldots,p_n}$ is a quasiarithmetic mean, condition (ii) implies (i). ■

Let us note that from the formulas (2) and (3) we obtain

$$\left(\mathscr{A}^{[\log\circ\gamma]}_{p_1,\ldots,p_n}\right)^{[\exp]} = \mathscr{A}^{[\log\circ\gamma\circ\exp]}_{p_1,\ldots,p_n}.$$

By analogy to the quasiarithmetic means, the mean $\mathscr{M}^{[\gamma]}_{p_1,\ldots,p_n}$ could be called a *weighted quasigeometric mean,* the function γ its *generator,* and $p_1, \ldots, p_n$, the weights.

Hence, applying Remark 3 and Proposition 2, we obtain the following characterization of weighted quasiarithmetic (or weighted quasigeometric) means on $(0, \infty)$ and satisfying condition (1).

Theorem 1 *Let $n \in \mathbb{N}$, $n \geq 2$, and let the positive numbers $p_1, \ldots, p_n$ such that $p_1 + \ldots + p_n = 1$ be fixed. Suppose that $\gamma : (0, \infty) \to (0, \infty)$ is continuous and strictly monotonic. Then the following conditions are equivalent:*

(i) the mean $\mathscr{M}^{[\gamma]}_{p_1,\dots,p_n} : (0,\infty) \to (0,\infty)$ *satisfies condition*

$$\mathscr{M}^{[\gamma]}_{p_1,\dots,p_n}(x_1,\dots,x_n)\,\mathscr{M}^{[\gamma]}_{p_1,\dots,p_n}\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right) = 1, \qquad x_1,\dots,x_n > 0;$$

(ii) the generator γ *satisfies the functional equation*

$$\gamma(x)\,\gamma\left(\frac{1}{x}\right) = [\gamma(1)]^2, \quad x > 0.$$

The main result of this paper reads as follows:

Theorem 2 *Let* $n \in \mathbb{N}$*,* $n \geq 2$*, and let the positive numbers* $p_1,\dots,p_n$ *be such that* $p_1 + \ldots + p_n = 1$*. Suppose that* $\gamma : (0,\infty) \to (0,\infty)$ *is continuous and strictly monotonic. Then the following conditions are equivalent:*

(i) the mean $\mathscr{M}^{[\gamma]}_{p_1,\dots,p_n} : (0,\infty)^n \to (0,\infty)$ *is homogeneous in* $(0,\infty)$ *and*

$$\mathscr{M}^{[\gamma]}_{p_1,\dots,p_n}(x_1,\dots,x_n)\,\mathscr{M}^{[\gamma]}_{p_1,\dots,p_n}\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right) = 1, \quad x_1,\dots,x_n > 0;$$

(ii) there are positive $a,b \in \mathbb{R}$*,* $a \neq 0$ *such that*

$$\gamma(t) = e^b t^a, \quad t > 0;$$

(iii) $\mathscr{M}^{[\gamma]}_{p_1,\dots,p_n} = \mathscr{G}_{p_1,\dots,p_n}$*, where*

$$\mathscr{G}_{p_1,\dots,p_n}(x_1,\dots,x_n) := x_1^{p_1} \cdot \ldots \cdot x_n^{p_n}, \quad x_1,\dots,x_n > 0,$$

is the weighted geometric mean. □

PROOF Let us assume (i). By Remark 6(ii) (or Remark 5), we have $\mathscr{M}^{[\gamma]}_{p_1,\dots,p_n} = \mathscr{A}^{[\log\circ\gamma]}_{p_1,\dots,p_n}$. The homogeneity of the weighted quasiarithmetic mean $\mathscr{A}^{[\log\circ\gamma]}_{p_1,\dots,p_n}$ implies (see [13, p. 68]) that there are $a,b,r \in \mathbb{R}$, $a \neq 0$ such that

$$\log\circ\gamma(t) = \begin{cases} at^r + b & \text{if} \quad r \neq 0 \\ a\log t + b & \text{if} \quad r = 0 \end{cases}, \quad t > 0.$$

If $r \neq 0$ then

$$\gamma(t) = e^b \exp(at^r), \quad t > 0,$$

whence

$$\gamma(t)\,\gamma\left(\frac{1}{t}\right) = e^b \exp(at^r)\, e^b \exp\left(a\left(\frac{1}{t}\right)^r\right) = e^{2b}\exp\left(a\left(t^r + t^{-r}\right)\right),$$

so the function $(0,\infty) \ni t \to \gamma(t)\,\gamma\left(\dfrac{1}{t}\right)$ is not constant.

If $r = 0$ then

$$\gamma(t) = e^b t^a, \quad t > 0,$$

whence

$$\gamma(t)\,\gamma\left(\frac{1}{t}\right) = \left(e^b t^a\right)\left(e^b t^{-a}\right) = \left(e^b\right)^2 = [\gamma(1)]^2, \quad t > 0.$$

Now (ii) follows from Theorem 1.

Assume (ii). Then

$$\gamma^{-1}(u) = e^{-\frac{b}{a}} u^{\frac{1}{a}}, \quad u \in \gamma(0,\infty),$$

and, taking into account that $p_1 + \ldots + p_n = 1$, we have, for all $x_1, \ldots, x_n > 0$,

$$\begin{aligned} \mathscr{M}^{[\gamma]}_{p_1,\ldots,p_n}(x_1,\ldots,x_n) &= \gamma^{-1}\left(\prod_{j=1}^{n}[\gamma(x_j)]^{p_j}\right) = e^{-\frac{b}{a}}\left(\prod_{j=1}^{n}\left(e^b x_j^a\right)^{p_j}\right)^{1/a} \\ &= \prod_{j=1}^{n} x_j^{p_j} = \mathscr{G}_{p_1,\ldots,p_n}(x_1,\ldots,x_n), \end{aligned}$$

so (iii) holds true.

Since the implication $(iii) \Longrightarrow (i)$ is obvious, the proof is complete. ∎

Remark 7 Let us note that the main results of this paper can be also extended to the class generalized weighted quasiarithmetic means of the form

$$\mathscr{M}^{[g_1,\ldots,g_n]}(x_1,\ldots,x_n) := \left(\prod_{j=1}^{n} g_j\right)^{-1}\left(\prod_{j=1}^{n} g_j(x_j)\right)$$

where $g_1, \ldots, g_n : (0,\infty) \to (0,\infty)$ are continuous strictly increasing (or strictly decreasing) functions and $\prod_{j=1}^{n} g_j$ is a single variable function being the product of $g_1 \ldots g_n$ (see [14]). Also in this case, if $\mathscr{M}^{[g_1,\ldots,g_n]}$ is homogeneous and satisfies (1), then $\mathscr{M}^{[g_1,\ldots,g_n]} = \mathscr{G}_{p_1,\ldots,p_n}$ for some positive $p_1, \ldots, p_n$ such that $p_1 + \ldots + p_n = 1$. □

We conclude this paper with the following

Remark 8 In the family of all homogeneous weighted quasiarithmetic means, only the geometric mean is self reciprocally conjugate, i.e. it satisfies condition (1). □

Let us also mention that (positive) homogeneity property plays in the principle of equivalent utility (see [17, 18]).

7. Conclusion

It has been shown that among a large family of all n-variable means (including the weighted quasiarithmetic means), the geometric mean is the most proper tool in the money exchange operations. Relation (1), the key property of the mean, can be interpreted as the equality of the mean, and its ”harmonic conjugate forces M to be the geometric mean. The classical arithmetic and harmonic means are important in applications. For instance, the harmonic mean in some situations provides the correct averages: of speed of a point, the density of a homogeneous mass formed with components of given masses and densities and is applied in electricity and finance. In this context the invariance of geometric mean with respect to the arithmetic-harmonic mean-type mapping gives hope for further applications of the geometric mean.

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