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THE GRADIENT AND THE DIVERGENCE FOR VECTOR-VALUED FORMS

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Abstract. On a Riemannian manifold, two differential operators: the gradient and the divergence are defined and investigated in the bundle of alternating differential forms of any degree with values in a vector bundle. Several algebraic, analytic and geometric properties of the two operators are derived. The vector character of the gradient on forms turns out to be a source of possible applications.

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1. Introduction

Rummler introduced in his paper [1] some new operators like grad, div, j, α, tr in the bundle of scalar differential forms on a Riemannian manifold M. He derived there many of their nice algebraic, analytic and geometric properties. In the present paper we extend the action of Rummler operators to the bundle of forms with values in a vector bundle. In particular, to the bundle of forms with values in the tangent bundle. Such forms will be called here *vector-valued forms* or shortly *vector forms*. We derive almost all the Rummler results in this more general setting. We concentrate mainly on two differential operators: grad and div as the operators that potentially can find applications in geometry, physics or engineering. With this in mind, we derive important properties of the operators. In particular, we prove Theorem 1 that the two operators are differentiations of the algebra of forms and Theorem 2 which states that the operators commute with the Hodge star operator. We also prove that the operators grad and -div are formally adjoint to each other with respect to the global scalar product (see: Theorem 4). The next properties of the operators, especially that from Lemma 5 and Lemma 6, lead in a direct way to Theorem 5 with the Weitzenböck formula establishing a relation between two *Laplacians* on vector valued forms: the Rummler Laplacian div grad and the Hodge-de Rham laplacian Δ. These two important Laplacians, though both being differential operators of order two, differ essentially only by an operator of order zero, i.e., by an endomorphism of the bundle the operators act on. Moreover, the shape of the endomorphism is explicitly given in formula (28). Its alternative version (30) emphasizes a dependence of the endomorphism on the curvature of the Riemannian manifold. This fact and the importance of the Hodge-de Rham Laplacian in geometry and physics is a source of many applications. Examples of such applications may be found eg., in [2] [3], or [4]. Recall that, in particular, by the standard Bochner technique we can get information on the existence such important deformations as isometric, projective, conformal or harmonic (cf. [5]). Moreover, one can also get estimates for the lower bounds for spectra of some differential operators like, eg., the Hodge-de Rham, Ahlfors or Lichnerowicz Laplacians (cf. [6], [7] or [8]).

Our aim is not only a generalization of the Rummler results but also a unification of his and ours results. We have managed to achieve that aim, by adopting suitable definitions for exterior products of different types of forms and for different types of exterior derivations.

A short review of some possible applications may be found in Section 6. Now, let us only mention that the derived here properties of the two differential operators: the gradient and the divergence enable formulating and proving an essential *generalization of the classical divergence theorem*. The generalization is dealing with the divergence of a vector form of any degree. It holds not only for geometrically flat spaces like the bounded euclidean domains, but also more generally, for Riemannian manifolds with the boundary. In this case, the Riemannian metric of a manifold enables emphasizing a possible substantial inhomogeneity of the considered domain (body). Some initial information on the mentioned subject may be found in [9]. The generalization and some of its applications in physics and engineering were also a subject of the lectures by the second named author at the Conferences on Mathematical Modeling in Physics and Engineering (MMPE'22 and MMPE'23). An extended and more detailed version completed with more advanced applications will be the subject of a subsequent paper.

2. Spaces of forms, exterior products, metrics and the Hodge star

For the notions of bundles of tensors and exterior forms and for the tensor and exterior products discussed in this section, we refer to [10].

Let M be an oriented Riemannian manifold, possibly also with a boundary, $\dim M = n$, with a scalar product (riemannian metric) $<,>=<,>_g$ in the tangent bundle T. The metric can naturally be extended to the cotangent bundle T^*. The extension will be denoted by the same symbol. Let $\Lambda^p = C^\infty(\Lambda^p T^*)$ be the space of alternating scalar p-forms on M. The exterior product $\wedge : \Lambda^p \times \Lambda^q \longrightarrow \Lambda^{p+q}$ of two such p- and q-forms at $x \in M$ is defined as follows:

$$(\varphi \wedge \psi)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = \\ \sum_{\sigma \in \mathrm{sh}(k,l)} \mathrm{sign}\sigma\, \varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)})\psi(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}). \tag{1}$$

for $v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l} \in T_x$ where $\mathrm{sh}(k,l)$ is the set of all shuffles of type (k,l).

The scalar product of two simple p-forms $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_p$ and $\psi = \psi_1 \wedge \cdots \wedge \psi_p$ is defined by the determinant:

$$\left\langle \varphi_1 \wedge \cdots \wedge \varphi_p, \psi_1 \wedge \cdots \wedge \psi_p \right\rangle_{\Lambda^p} = \sum_{\sigma \in S_p} \mathrm{sign}\sigma \left\langle \varphi_1, \psi_{\sigma_1} \right\rangle_g \cdots \langle \varphi_p, \psi_{\sigma_p} \rangle_g, \tag{2}$$

where $\varphi_1, \dots, \varphi_p, \psi_1, \dots, \psi_p \in \Lambda^1 = {T^*}^1$, and then extended to the space of all p-forms be linearity.

All manifolds and mappings are assumed to be smooth, i.e. of class C^∞. For any bundle E on M, we will often denote $C^\infty(E)$ – the space of sections of E simply by E. Let E be a vector bundle on M with a scalar product $<,>_E$. Consider the following spaces of forms:
$\vec{\Lambda}^p = C^\infty(\Lambda^p T^* \otimes T)$ – the space of vector p-forms
$\Lambda^p(E) = C^\infty(\Lambda^p T^* \otimes E)$ – the space of scalar p-forms with values in E
$\vec{\Lambda}^p(E) = C^\infty(\Lambda^p T^* \otimes E \otimes T)$ – the space of vector p-forms with values in E.

Define exterior products (all denoted by the same symbol $\wedge$) for all the possible pairs of forms:

$$\wedge : \begin{cases} \Lambda^p \times \vec{\Lambda}^q \longrightarrow \vec{\Lambda}^{p+q} \\ \Lambda^p \times \Lambda^q(E) \longrightarrow \Lambda^{p+q}(E) \\ \Lambda^p \times \vec{\Lambda}^q(E) \longrightarrow \vec{\Lambda}^{p+q}(E) \\ \vec{\Lambda}^p \times \vec{\Lambda}^q \longrightarrow \Lambda^{p+q} \\ \vec{\Lambda}^p \times \Lambda^q(E) \longrightarrow \vec{\Lambda}^{p+q}(E) \\ \vec{\Lambda}^p \times \vec{\Lambda}^q(E) \longrightarrow \Lambda^{p+q}(E) \\ \Lambda^p(E) \times \Lambda^q(E) \longrightarrow \Lambda^{p+q} \\ \Lambda^p(E) \times \vec{\Lambda}^q(E) \longrightarrow \vec{\Lambda}^{p+q} \\ \vec{\Lambda}^p(E) \times \vec{\Lambda}^q(E) \longrightarrow \Lambda^{p+q} \end{cases}$$

according to the following examples that enable the natural understanding of the exterior product in all the remaining cases:

$$\omega \wedge (\eta \otimes Y) = \omega \wedge \eta \otimes Y$$
$$(\omega \otimes X) \wedge (\eta \otimes Y) = \omega \wedge \eta \cdot \left\langle X, Y \right\rangle_g$$
$$(\omega \otimes s) \wedge (\eta \otimes t) = \omega \wedge \eta \cdot \left\langle s, t \right\rangle_E$$
$$(\omega \otimes s) \wedge (\eta \otimes t \otimes Y) = \omega \wedge \eta \cdot \left\langle s, t \right\rangle_E \otimes Y$$

$$(\omega \otimes s \otimes X) \wedge (\eta \otimes t \otimes Y) = \omega \wedge \eta \cdot \langle s,t \rangle_E \cdot \langle X,Y \rangle_g.$$

The accepted definitions have this advantage that they unify all the possible actions in the sets of scalar and vector forms and such forms with values in E within the only symbol $\wedge$. The scalar products in T, Λ^p and E define the natural scalar products in $\vec{\Lambda}^p = \Lambda^p \otimes T$, $\Lambda^p(E) = \Lambda^p \otimes E$, $\vec{\Lambda}(E) = \Lambda^p \otimes E \otimes T$ by

$$\langle \omega \otimes X, \eta \otimes Y \rangle = \langle \omega, \eta \rangle_{\Lambda^p} \cdot \langle X,Y \rangle_g,$$
$$\langle \omega \otimes s, \eta \otimes t \rangle = \langle \omega, \eta \rangle_{\Lambda^p} \cdot \langle s,t \rangle_E$$
$$\langle \omega \otimes s \otimes X, \eta \otimes t \otimes Y \rangle = \langle \omega, \eta \rangle_{\Lambda^p} \cdot \langle s,t \rangle_E \cdot \langle X,Y \rangle_g,$$

respectively.

From now on all the scalar products will be denoted simply by $\langle , \rangle$.

The Riemannian structure and the orientation define on M a unique form Ω of the maximal degree n characterized by the condition $\Omega_M(e_1,\dots,e_n) = 1$ for any local positively oriented orthonormal frame of vectors $e_1,\dots,e_n$ on M. By taking the dual frame of 1-forms: $e_1^*,\dots,e_n^*$, we get easily, that locally,

$$\Omega_M = e_1^* \wedge \ldots \wedge e_n^*.$$

Recall (cf. [10]) that the classical *Hodge star* is the linear operator, $\star : \Lambda^p \longrightarrow \Lambda^{n-p}$, defined by

$$\varphi \wedge \star\psi = \langle \varphi, \psi \rangle \Omega_M \tag{3}$$

for $\varphi, \psi \in \Lambda^p$.

One can calculate that then, in local oriented orthogonal frame $e_1^*,\dots,e_n^*$,

$$\star(e_{i_1}^\star \wedge \ldots \wedge e_{i_p}^*) = \mathrm{sgn}(i_1,\dots,i_p,j_1,\dots,j_{n-p})(e_{j_1}^* \wedge \ldots \wedge e_{j_{n-p}}^*)$$

for any permutation $i_1,\dots,i_p,j_1,\dots,j_{n-p}$ of $1,\dots,n$. This easily implies that

$$\star\star \mid_{\Lambda^p} = (-1)^{p(n-p)} \mathrm{id} \mid_{\Lambda^p} . \tag{4}$$

3. Differential operators

Let ∇ be the Levi-Civita connection in the tangent bundle T,

$$\nabla : T \longrightarrow T^* \otimes T$$

i.e., the unique connection in T which is *metric* and *torsion free*, i.e. the following two conditions: $\nabla\langle s,t \rangle_g = \langle \nabla s,t \rangle_g + \langle s, \nabla t \rangle_g$ and $\nabla_X Y - \nabla_Y X - [X,Y] = 0$ are satisfied for all vector fields X and Y(cf. [2] or [3]).

The connection ∇ can be extended naturally to the cotangent bundle, and next to any tensor bundle on M by the Leibniz rule and finally to any of its subbundle so, in particular, to the bundle Λ^p, for $p = 1,2,\dots$.

Assume also, that the bundle E is equipped with a metric connection

$$\nabla^E : E \longrightarrow T^* \otimes E,$$

i.e. such connection that $\nabla^E \langle s,t \rangle_E = \langle \nabla^E s,t \rangle_E + \langle s, \nabla^E t \rangle_E$.

Let: $\mathrm{d} : \Lambda^p \longrightarrow \Lambda^{p+1}$ be the standard operator of derivation of scalar exterior forms. Define three next derivation operators:

– on vector p-forms, $\vec{\mathrm{d}} : \vec{\Lambda}^p \longrightarrow \vec{\Lambda}^{p+1}$:

$$\vec{\mathrm{d}}(\omega \otimes X) = \mathrm{d}\omega \otimes X + (-1)^p \omega \wedge \nabla X,$$

– on scalar p-forms with values in E, $\mathrm{d}^\nabla : \Lambda^p(E) \longrightarrow \Lambda^{p+1}(E)$:

$$\mathrm{d}^E(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^p \omega \wedge \nabla^E s,$$

– on vector p-forms with values in E, $\vec{\mathrm{d}}^E : \vec{\Lambda}^p(E) \longrightarrow \vec{\Lambda}^{p+1}(E)$:

$$\vec{\mathrm{d}}^E(\omega \otimes s \otimes X) = \mathrm{d}\omega \otimes s \otimes X + (-1)^p \omega \wedge (\nabla^E s \otimes X + s \otimes \nabla X).$$

Note that ∇X and $\nabla^E s$ are treated here as one-forms with values in the bundles T and E, respectively. For simplicity, all the exterior derivations $\mathrm{d}, \vec{\mathrm{d}}, \mathrm{d}^E, \vec{\mathrm{d}}^E$ will be denoted by the same letter d, all the connections ∇, ∇^E by the same symbol ∇. By standard calculations we derive that for every $\xi \in \Lambda^p$ (or $\vec{\Lambda}^p$ or $\vec{\Lambda}^p(E)$) and $\eta \in \Lambda^q$ (or $\vec{\Lambda}^q$ or $\vec{\Lambda}^q(E)$) we have – in each case – the same rule saying that d is antiderivation:

$$\mathrm{d}(\xi \wedge \eta) = \mathrm{d}\xi \wedge \eta + (-1)^p \xi \wedge \mathrm{d}\eta. \tag{5}$$

Example 1 If, near $x \in M$, $e_1^*, \ldots, e_n^*$ is a local orthonormal frame normal at x in the sense that $\nabla e_i^* = 0,\ i = 1, \ldots, n$, then, at x, $\mathrm{d}\big(e_{i_1}^* \wedge \ldots \wedge e_{i_p}^*\big) = 0$. So, by the definition of derivation operator d,

$$\mathrm{d}\big(e_{i_1}^* \wedge \ldots \wedge e_{i_p}^* \otimes s\big) = (-1)^p e_{i_1}^* \wedge \ldots \wedge e_{i_p}^* \wedge \nabla s. \tag{6}$$

CONVENTION 1 Here and afterwards, $e_1, \ldots, e_n \in T$ is an local orthonormal frame, and $e_1^*, \ldots, e_n^* \in T^*$ is the dual one.

Local expressions within these frames give us better understanding of the action of the considered operators.

Proposition 1 *The exterior derivation has the following local form:*

$$\mathrm{d} = \sum_{j=1}^{n} e_j^* \wedge \nabla_{e_j}. \tag{7}$$

PROOF Just a calculation in local orthonormal frames $e_1, \ldots, e_n \in T$ and $e_1^*, \ldots, e_n^* \in T^*$. See also ([11], Lemma 1.4.3). ∎

Now (in analogy to [1]), define two operators of order zero that will be used in the construction of the two main operators: the gradient and the divergence.

Definition 1 $\mathrm{j} : \Lambda^p(E) \longrightarrow \vec{\Lambda}^{p-1}(E)$ is the linear operator defined locally on simple forms by:

$$\mathrm{j}(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s) = \sum_{k=1}^{p} (-1)^{k-1} e^*_{i_1} \wedge \ldots \wedge \hat{e^*}_{i_k} \ldots \wedge e^*_{i_p} \otimes s \otimes e_{i_k}. \tag{8}$$

Lemma 1 j *is antiderivation i.e. for any* $\xi \in \Lambda^p(E)$, $\eta \in \Lambda^q(E)$

$$\mathrm{j}(\xi \wedge \eta) = \mathrm{j}\xi \wedge \eta + (-1)^p \xi \wedge \mathrm{j}\eta.$$

PROOF Without loss of generality, we check the formula for simple tensors of the form: $\xi = e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s$ and $\eta = e^*_{j_1} \wedge \ldots \wedge e^*_{j_q} \otimes t$. And this is just a calculation.■

The operator j commutes with the covariant derivative ∇ in the following sense:

Lemma 2 *For every* $X \in T$ *we have* $\nabla_X \mathrm{j} = \mathrm{j}\nabla_X$ □

PROOF Let $x \in M$ and $e_1, \ldots, e_n \in T$, $e^*_1, \ldots, e^*_n \in T^*$ be dual local orthonormal frames, normal at x ($\nabla e_j = 0$, $\nabla e^*_j = 0$ at x). Then:

$$\begin{aligned} &\nabla_X \mathrm{j}(e^*_{i_1} \wedge \ldots e^*_{i_p} \otimes s) = \nabla_X \left(\sum_{k=1}^{p} (-1)^{k-1} e^*_{i_1} \wedge \ldots \hat{e^*_{i_k}} \wedge e^*_{i_p} \otimes s \otimes e_{i_k} \right) \\ &= \sum_{k=1}^{p} (-1)^p e^*_{i_1} \wedge \ldots \hat{e^*_{i_k}} \wedge e^*_{i_p} \otimes \nabla_X s \otimes e_{i_k} = \mathrm{j}\nabla_X \left(e^*_{i_1} \wedge \ldots e^*_{i_p} \otimes s \right) \text{ at } x. \end{aligned}$$

Definition 2 $\alpha : \vec{\Lambda}^p(E) \longrightarrow \Lambda^{p+1}(E)$ is the linear operator defined locally on simple tensors by:

$$\alpha(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s \otimes e_j) = e^*_j \wedge e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s \tag{9}$$

Unfortunately, α has no property similar to that from Lemma 1. We can prove yet that $\alpha\mid_{\vec{\Lambda}^p(E)}$ and $\mathrm{j}\mid_{\Lambda^{p+1}(E)}$ are adjoint to each other:

Lemma 3 *For* $\varphi \in \vec{\Lambda}^p(E)$ *(or* $\vec{\Lambda}^p$*) and* $\psi \in \Lambda(E)$ (or Λ^p)

$$\langle \alpha(\varphi), \psi \rangle = \langle \varphi, \mathrm{j}(\psi) \rangle \tag{10}$$

PROOF Let $i_1 < \ldots < i_p$. If $\mathrm{j} \notin \{i_1, \ldots, i_p\}$ then

$$\begin{aligned} &\langle \alpha(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s \otimes e_j), e^*_j \wedge e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes t \rangle \\ =&\langle e^*_j \wedge e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s, e^*_j \wedge e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes t \rangle = \langle s, t \rangle. \end{aligned}$$

On the other hand

$$\begin{aligned}
&\left\langle e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes s\otimes e_j, j(e^*_j\wedge e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes t)\right\rangle\\
&=\left\langle e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes s\otimes e_j, e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes t\otimes e_j\right\rangle\\
&-\left\langle e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes s\otimes e_j, \sum_{k=1}^{p}(-1)^{k-1}e^*_j\wedge e^*_{i_1}\wedge\ldots\wedge \hat{e^*}_{i_k}\ldots\wedge e^*_{i_p}\otimes s\otimes e_{i_k}\right\rangle\\
&=\langle s,t\rangle - 0 = \langle s,t\rangle,
\end{aligned}$$

so, the both sides of (10) are equal. If j $\in\{i_1,\ldots,i_p\}$ then the both sides are equal to zero. ∎

The operator α commutes with ∇ in the following sense:

Lemma 4 *For every $X\in T$ we have:*

$$\nabla_X\alpha = \alpha\nabla_X$$

PROOF Similar to that of Lemma 2. ∎

Finally define in a standard way the trace operator.

Definition 3 The *trace opertor*, tr : $\vec{\Lambda}^p(E)\longrightarrow\Lambda^{p-1}(E)$, is defined by:

$$\mathrm{tr}\,\Phi(e_{i_1},\ldots,e_{i_{p-1}}) = \sum_{i=1}^{n}\left\langle\Phi(e_i,e_{i_1},\ldots,e_{i_{p-1}}),e_i\right\rangle \tag{11}$$

where $e_1,\ldots,e_n\in T$ is, as usual, a local orthonormal frame. □

4. The gradient and the divergence

In this section, two first order linear differential operators grad and div, will be introduced. The first one acts on exterior forms with values in a bundle E and generalizes the classical gradient acting on functions, the other one acts on vector exterior forms with values in a bundle E and generalizes the classical divergence acting on vector fields.

Definition 4 The *gradient* is the differential operator, grad : $\Lambda^p(E)\longrightarrow\vec{\Lambda}^p(E)$, defined by:

$$\mathrm{grad} = \mathrm{j\,d} + \mathrm{d\,j}. \tag{12}$$

Example 2 If, near $x\in M$, $\xi = e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes s$ for a local orthonormal frame $e^*_1,\ldots,e^*_n$ normal at x in the sense that $\nabla e^*_i = 0, i=1,\ldots,n$, then, at x,

$$\mathrm{grad}\,\xi = e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes(\nabla s)^\flat \tag{13}$$

where, for any one form ω (including forms with values in a vector bundle), $\omega^\flat$ is defined by:

$$\omega(X) = < \omega^\flat, X >_g, \quad X \in T_x M. \tag{14}$$

If, in particular,

$$\xi = f\, e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \tag{15}$$

for some function f, then

$$\operatorname{grad} \xi = e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes \operatorname{grad} f \quad \text{at } x. \tag{16}$$

PROOF Just a calculation in the local orthonormal frame normal at x. ■

Definition 5 The *divergence* is the differential operator, div : $\vec{\Lambda}^p(E) \longrightarrow \Lambda^p(E)$, defined by:

$$\operatorname{div} = \operatorname{tr} \mathrm{d} + \mathrm{d} \operatorname{tr}. \tag{17}$$

Example 3 Similarly, as in Example 2, we have, in a local orthonormal frame normal at x, that

$$\operatorname{div}(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s \otimes e_j) = e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes \nabla_{e_j} s \quad \text{at } x. \tag{18}$$

Theorem 1 *The operators* grad *and* div *satisfy the following relations:*

a) $\xi \in \Lambda^p(E), \eta \in \Lambda^p(E) \Longrightarrow \operatorname{grad}(\xi \wedge \eta) = (\operatorname{grad} \xi) \wedge \eta + \xi \wedge \operatorname{grad} \eta$,

b) $\xi \in \Lambda^p(E), \eta \in \vec{\Lambda}^p(E) \Longrightarrow \operatorname{div}(\xi \wedge \eta) = (\operatorname{grad} \xi) \wedge \eta + \xi \wedge \operatorname{div} \eta$. □

PROOF

a) Let $\xi = e^*_{i1} \wedge \ldots \wedge e^*_{ip} \otimes s$ and $\eta = e^*_{j1} \wedge \ldots \wedge e^*_{jq} \otimes t$. By (13) and (3) we have sequentially:

$$\begin{aligned}
&\operatorname{grad}(\xi \wedge \eta) = \operatorname{grad}(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \wedge e^*_{j_1} \wedge \ldots \wedge e^*_{j_q}) \cdot \langle s, t \rangle) \\
&= (e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \wedge e^*_{j_1} \wedge \ldots \wedge e^*_{j_q}) \otimes (\nabla \langle s, t \rangle)^\flat \\
&= (e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \wedge e^*_{j_1} \wedge \ldots \wedge e^*_{j_q}) \otimes \langle (\nabla s)^\flat, t \rangle \\
&+ (e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \wedge e^*_{j_1} \wedge \ldots \wedge e^*_{j_q}) \otimes \langle s, (\nabla t)^\flat \rangle \\
&= \left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes (\nabla s)^\flat \right) \wedge \left(e^*_{j_1} \wedge \ldots \wedge e^*_{j_q} \otimes t \right) \\
&\quad + \left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s \right) \wedge \left(e^*_{j_1} \wedge \ldots \wedge e^*_{j_q} \otimes (\nabla t)^\flat \right) = (\operatorname{grad} \xi) \wedge \eta + \xi \wedge (\operatorname{grad} \eta).
\end{aligned}$$

The proof of b) is similar. ■

Extend the classical Hodge star operator defined by (3) to the bundle of forms with values in E and to the bundle of vector forms with values in E:

The *Hodge star operators* $\star : \Lambda^p(E) \longrightarrow \Lambda^{n-p}(E)$ and $\star : \vec{\Lambda}^p(E) \longrightarrow \vec{\Lambda}^{n-p}(E)$ are defined by:

$$\star\big(\varphi \otimes s\big) = \big(\star\varphi\big) \otimes s \tag{19}$$

and

$$\star\big(\varphi \otimes s \otimes X\big) = \big(\star\varphi\big) \otimes s \otimes X, \tag{20}$$

respectively.

Theorem 2 *The operators* grad *and* div *commute with the Hodge star:*

a) $\star\mathrm{grad} = \mathrm{grad}\star$,

b) $\star\mathrm{div} = \mathrm{div}\star$. □

PROOF

a) By (13) and the definition of the Hodge star,

$$\star\mathrm{grad}\left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s\right) = \star\left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes (\nabla s)^\flat\right) = \star\left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}\right) \otimes (\nabla s)^\flat.$$

Similarly,

$$\mathrm{grad}\star\left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p} \otimes s\right) = \mathrm{grad}\left(\star\left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}\right) \otimes s\right) = \star\left(e^*_{i_1} \wedge \ldots \wedge e^*_{i_p}\right) \otimes (\nabla s)^\flat.$$

The proof of b) is similar. ■

Theorem 3 *The operators* grad *and* $-$div *are formally adjoint, i.e. for* $\xi \in \Lambda^p(E)$ *and* $\eta \in \vec{\Lambda}^p(E)$*:*

$$\big(\mathrm{grad}\,\xi, \eta\big) = \big(\xi, -\mathrm{div}\eta\big),$$

if only ξ *or* η *is of compact support not intersecting the boundary.*

Here

$$\big(\zeta, \psi\big) := \int_M \zeta \wedge \star\psi, \tag{21}$$

for $\zeta, \psi \in \Lambda^q(E)$ *(or* $\zeta, \psi \in \vec{\Lambda}^q(E)$*).* □

PROOF For $\xi \in \Lambda^p(E)$, $\eta \in \vec{\Lambda}^p(E)$ the exterior form $\xi \wedge \star\eta$ is of the maximal degree, so, by the definition of divergence,

$$\mathrm{div}(\xi \wedge \star\eta) = \mathrm{d}\,\mathrm{tr}(\xi \wedge \star\eta). \tag{22}$$

On the other hand, by Theorem 1 b) and by Theorem 2 b), we get

$$\operatorname{div}(\xi \wedge \star\eta) = \operatorname{grad}\xi \wedge \star\eta + \xi \wedge \star \operatorname{div}\eta,$$

so, by (22),

$$\mathrm{d}\,\mathrm{tr}\,(\xi \wedge \star\eta) = \operatorname{grad}\xi \wedge \star\eta + \xi \wedge \star \operatorname{div}\eta.$$

Integrating over M, applying the Stokes theorem and using the compactness of supports and the assumptions thet that they are not intersecting the boundary, we get the assertion. ■

Definition 6 The differential operator $\delta : \Lambda^p(E) \longrightarrow \Lambda^{p-1}(E)$ is defined by

$$\delta = (-1)^{np+n} \star \mathrm{d} \star. \tag{23}$$

Remark 1 Note that δ differs here by sign from the codifferential that occurs usually in differential geometry and denoted there by the same letter (cf. [2] or [3]). □

Theorem 4 *The operators* d *and* $-\delta$ *are formally adjoint, i.e. for* $\xi \in \Lambda^p(E)$, $\eta \in \Lambda^{p+1}(E)$ *(or* $\xi \in \vec{\Lambda}^p(E)$, $\eta \in \vec{\Lambda}^{p+1}(E)$*):*

$$\big(\mathrm{d}\xi, \eta\big) = \big(\xi, -\delta\eta\big),$$

if only ξ *or* η *is of a compact support not intersecting the boundary.* □

PROOF By the rule of differentiation of the wedge product (cf. (5)) we have

$$\mathrm{d}(\xi \wedge \star\eta) = \mathrm{d}\xi \wedge \star\eta + (-1)^p \xi \wedge \mathrm{d}\star\eta.$$

Integrating over M and using the Stokes theorem, we get

$$0 = \int_M \mathrm{d}\xi \wedge \star\eta + \int_M (-1)^p \xi \wedge \mathrm{d}\star\eta.$$

Applying (4) adapted to the form $\mathrm{d}\star\eta$ of degree $n-p$, we get

$$\int_M \mathrm{d}\xi \wedge \star\eta = -\int_M (-1)^p \xi \wedge (-1)^{p(n-p)} \star\star \mathrm{d}\star\eta$$

or

$$\int_M \mathrm{d}\xi \wedge \star\eta = -\int_M \xi \wedge \star\big((-1)^{pn} \star \mathrm{d}\star\eta\big).$$

Now, by (23) adapted to the action on a form of the degree $p+1$ and by (21), we get the assertion. ■

Lemma 5 *The following relations between the operators* d, δ *and* grad *hold:*

a) $\mathrm{d} = \alpha\,\mathrm{grad}$, *b)* $\delta = \operatorname{tr}\operatorname{grad}$. □

PROOF a)

$$\alpha \operatorname{grad}\big(e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes s\big) = \alpha\left(e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes(\nabla s)^\flat\right)$$
$$= \alpha\big(e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes\sum_{j=1}^n e_j\otimes s_j\big) = \sum_{j=1}^n e^*_j\wedge e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes s_j,$$

for some sections s_j of E. On the other hand, by (6),

$$\mathrm{d}\big(e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes s\big) = (-1)^p e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\wedge(\nabla s)$$
$$= (-1)^p e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\wedge\sum_{j=1}^n e^*_j\otimes s_j = \sum_{j=1}^n e^*_j\wedge e^*_{i_1}\wedge\ldots\wedge e^*_{i_p}\otimes s_j.$$

The proof of b) is similar. ■

5. The Weitzenböck formula

The Weitzenböck formula establishes a relation between two *Laplacians* in the bundle of forms with values in a vector bundle: the Rummler Laplacian div grad and the Hodge-de Rham Laplacian Δ. These two important Laplacians though both being differential operators of the second order differ essentially only by a zero order term, i.e., by an endomorphism of the bundle the operators act on. The most interesting is that this zero order operator (tensor) depends essentially on the geometry of the Riemannian manifold M or, more precisely, on the curvature tensor R of the Riemannian metric g.

Recall (cf. [11]) that the *Hodge-de Rham Laplacian* on forms with values in a vector bundle is the second order linear differential operator defined as follows:

$$\Delta = \delta\,\mathrm{d} + \mathrm{d}\,\delta. \tag{24}$$

Remark 2 Note that Δ differs here by sign from the the Hodge-de Rham Laplacian that usually occur in differential geometry (see also Remark 1). □

Recall (cf. [2] or [3]) that for $X,Y\in T$ the *curvature* R_{XY} of ∇ in directions X,Y is defined by

$$R_{XY} = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}. \tag{25}$$

It is known that the curvature operator is in fact an operator of order zero, so it can be regarded as an endomorphism of the bundle on which it acts. R and its trace $\mathscr{R}$ belong to the most important operators in differential geometry. The other will appear in the geometric version of the Weitzenböck formula below.

Definition 7 The operator $\mathscr{R}$, called the *Ricci endomorphism*, is defined locally by the curvature tensor R as follows:

$$\mathscr{R} = \sum_{i,j} e_j^* \wedge \iota_{e_i} R_{e_j e_i}. \tag{26}$$

Here ι_{e_i} denotes the substitution of the vector e_i on the place of first argument. □

Crucial for the proof of the Weitzenböck formula is the following fact:

Lemma 6

$$\operatorname{tr} \mathrm{j} = 0 \quad \text{and} \quad \delta = \operatorname{tr} \mathrm{d} \mathrm{j}. \tag{27}$$

PROOF For $\varphi \in \Lambda^p(E)$ we have by (8) and (11) that

$$\begin{aligned}(\operatorname{tr} \mathrm{j})\, \varphi(e_{i_1}, \ldots, e_{i_{p-1}}) &= \sum_{i=1}^{n} \left\langle (\mathrm{j}\, \varphi)(e_i, e_{i_1}, \ldots, e_{i_{p-1}}), e_i \right\rangle \\ &= \sum_{i=1}^{n} (\iota_{e_i} \varphi)(e_i, e_{i_1}, \ldots, e_{i_{p-1}}) = 0\end{aligned}$$

and

$$\delta = \operatorname{tr} \operatorname{grad} = \operatorname{tr}(\mathrm{d}\mathrm{j} + \mathrm{j}\mathrm{d}) = \operatorname{tr} \mathrm{d}\mathrm{j} + \underbrace{\operatorname{tr} \mathrm{j}\mathrm{d}}_{0} = \operatorname{tr} \mathrm{d}\mathrm{j}.$$

Now we are ready to state:

Theorem 5 (Weitzenböck Formula)

$$\operatorname{div} \operatorname{grad} = \Delta + \operatorname{tr} \mathrm{d}^2 \mathrm{j}\ . \tag{28}$$

PROOF By b) of lemma 5 and (27) we have sequentially

$$\operatorname{div} \operatorname{grad} = (\operatorname{tr} \mathrm{d} + \mathrm{d} \operatorname{tr})(\mathrm{j}\mathrm{d} + \mathrm{d}\mathrm{j}) = \underbrace{\operatorname{tr} \mathrm{d}\mathrm{j}\mathrm{d}}_{\delta \mathrm{d}} + \operatorname{tr} \mathrm{d}^2 \mathrm{j} + \underbrace{\mathrm{d} \operatorname{tr} \mathrm{j}\mathrm{d}}_{0} + \underbrace{\operatorname{tr} \mathrm{d} \operatorname{tr} \mathrm{d}\mathrm{j}}_{\mathrm{d}\delta} = \Delta + \operatorname{tr} \mathrm{d}^2 \mathrm{j},$$

so, the proof is completed. ■

Note that, in the light of formula (28), to establish the announced relation between the operator of form: $\operatorname{div} \operatorname{grad} - \Delta$ and the curvature operator R, it is enough to establish the relation between the operators d^2 and R. We have namely:

Proposition 2 *For $\varphi \in \vec{\Lambda}^p(E)$ (or $\vec{\Lambda}^p$)*

$$\mathrm{d}^2 \varphi = \varphi \wedge R. \tag{29}$$

PROOF A standard calculation with the use of local definitions (7) and (25) of the exterior derivative d and the curvature operator R, respectively. ∎

Now, by using (29) and (8), we can easily confirm that the last summand in (28) is just the Ricci endomorphism defined in (26). The Weitzeböck formula can then be written in its alternative (in fact more geometric) shape:

$$\operatorname{div}\operatorname{grad} = \Delta + \mathscr{R}. \tag{30}$$

Since $\mathscr{R}$ depends on R, formula (30) exposes in fact the dependence on curvature tensor R, i.e., on the geometry of M.

6. Conclusions and possible applications

The properties of the two operators investigated in the present paper: the gradient and the divergence enable formulating and proving an extension of the classical divergence theorem (one of the most important results of modern calculus with a wide spectrum of applications) onto vector valued forms. Its importance – especially when possible applications are considered – comes from the fact that it deals with a vector field. Vector fields represent namely a displacement of forces acting on a physical body. In engineering practice, the values of the forces can only be measured at the boundary. The divergence theorem gives then some information on what is going inside the body [9, 12].

Moreover, the vector character of the forces enables setting up systems of non-trivial boundary conditions that can be investigated when solving the boundary value problems. For example, in the theory of elastic body, the following four natural boundary conditions: Dirichlet, Absolute, Relative and Neumann are considered (cf. [13, 14]).

The classic gradient of a scalar function is a vector field with the property that in each particular point of the domain of definition of the function, the vector of the field points out the direction of the maximal growth of the function. This property is principal for many possible engineering applications especially for the approximate methods called *the gradient-based methods* that are very efficient when optimum or extreme points are to be detected. For a review of the methods we refer to [15, 16].

Let us note that similar methods will also be applicable in the case of gradient of forms of any degree considered here, and even in a more general geometric situation. Indeed, we can namely consider a Riemannian manifold D of dimension n, i.e., a region (domain) D equipped with a Riemannian scalar product (metric) that depends essentially on the points of D (for the theory of Riemannian manifolds see eg. [2] or [3]). In an engineering practice, the domain will be representing a physical body. The Riemannian metric, considered instead of the Euclidean one, will be representing then an inhomogeneity of the material of the body. Let us assume that D is foliated by p-dimensional submanifolds called the *leaves* of the *foliation* (for the theory of foliations see eg. [17]). Visual examples of foliations are often supplied by the nature. For instance, some territories (regions) of rocks in mountains are foliated by their geologic layers. These layers play then the role of the leaves of foliation. It is known that the Riemannian metric in D defines on each leaf of the foliation its unique volume form. These volume forms glue together to a global form of degree p in D. Under some additional assumptions on the foliation, this globally defined form, call it ξ, can locally be written in shape (15). But then at any point $x \in D$, the gradient of ξ defines a unique vector tangent to D at x. In analogy to the gradient of a function, this vector points out the direction of maximal growth for the norm of ξ. This norm is related to the displacement of the p-dimensional measure (mass) in D. Then, in an analogy to the classic gradient methods, we will be able to detect – by approximate methods – the points of local extreme condensation for the distribution of mass density in D.

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References

[1] Rummler, H. (1989). Differential forms. Weitzenböck formulae and foliations. *Publicacions Matematiques*, *33*, 543-554.

[2] Abraham, R., & Marsden, J. (1987). *Foundation of Mechanics*. Addison-Wesley Publishing Company, lnc.

[3] Nakahara, M. (2003). *Geometry, Topology and Physics*. Institute of Physics Publishing.

[4] Heil, K., Moroianu, A., & Semmelmann, U. (2016). Killing and conformal Killing tensors. *Geom. Phys.*, *106*, 383-400.

[5] Yano, K. (1970). *Integral Formulas in Riemannian Geometry*. Marcel Dekker, Inc.

[6] Gallot, S., & Meyer, D. (1975). Operateur de courbure et Laplacien des formes différentielles d'une variété riemannienne. *Math. Pures Appl.*, 259-284.

[7] Pierzchalski, A. (1989). Ricci curvature and quasiconformal deformations of a Riemannian manifold. *Manuscripta Mathematica*, *66*, 113-127.

[8] Boucetta, M. (2009). Spectra and symmetric eigentensors of the Lichnerowicz Laplacian on S^n. *Osaka J. Math.*, *46*, 239-254.

[9] Pierzchalski, A. (2021). Volume form, vector product and the divergence theorem. *Mathematical Modeling in Physics and Engeenering, Czestochowa 2021, Book of abstracts* 21-24. https://km.pcz.pl/konferencja/dokumenty/MMPE21-abstract-book.pdf

[10] Federer, H. (2014). *Geometric Measure Theory*. Grundlehren Math. Wiss., Springer.

[11] Yu, Y. (2001). *The Index Theorem and the Heat Equation Method*. World Scientific.

[12] Spivak M. (2018). *Calculus on Manifolds, a Modern Approach to Classical Theorems of Advanced Calculus*. CRC Press.

[13] Kozłowski, W., & Pierzchalski, A. (2008). Natural boundary value problems for weighted form Laplacians. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, *5*(7), 343-367.

[14] Pierzchalski, A. (2017). Gradients: the ellipticity and the elliptic boundary conditions – a jigsaw puzzle. *Folia Mathematica*, *19*, 65-83.

[15] Snyman, J. (2005). *Practical Mathematical Optimization: An Introduction to Basic Optimization Theory and Classical and New Gradient-based Algorithms*. Springer.

[16] Rao, S. (2009). *Engineering Optimization: Theory and Practice: Theory and Applications*. John Wiley and Sons.

[17] Candel, A., & Conlon, L. (2000), *Foliations I. Graduate Texts in Mathematics*. Providence, Rhode Island: AMS.