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SOME CLOSED FORM SERIES SOLUTIONS FOR THE TIME-FRACTIONAL DIFFUSION-WAVE EQUATION IN POLAR COORDINATES WITH A GENERALIZED CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper, we obtain some closed form series solutions for the time fractional diffusion-wave equation (TFDWE) with the generalized time-fractional Caputo derivative (GTFCD) associated with a source term in polar coordinates. These solutions are found using generalized Laplace and Hankel transforms. We obtained the closed form series solutions in the form of the Polygamma function. The effect of the fractional order derivative on the diffusion-wave variable is illustrated graphically.

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1. Introduction

Fractional differential equations (FDEs) have been applied to an increasing number of fields such as physics, engineering, and other sciences [1-7]. The time--fractional diffusion models have been used in various fields like biology, physics, chemistry, and finance. The time fractional diffusion equations preferably possess advantages for describing anomalous diffusion phenomena due to the memory property of fractional order derivatives [2-7]. Whereas the TFDWEs can be used to model the propagation of diffused waves in viscoelastic media [8].

The classical diffusion and wave equation is defined as [8]

$$\frac{dT}{dt} = a\,\Delta\,T, \quad \frac{d^2T}{dt^2} = a\,\Delta\,T.$$

So, the TFDWE can be obtained from the classical diffusion or wave equation by replacing the first or the second order time derivative with a fractional derivative of order $0 < \alpha < 2$ as follows:

$$\frac{\partial^{\alpha} T}{\partial t^{\alpha}} = a \, \Delta \, T, \;\; 0 < \alpha < 2. \tag{1}$$

These equations describe many important physical phenomena in different fields [9, 10]. Therefore, it has attracted the interest of many researchers in investigating solutions of these types of equations [11-21].

Investigating intermediate processes between diffusion and wave propagation modeled by the generalized fractional derivatives is considered as a hot topic in fractional calculus [22]. So, in this paper, we investigate the TFDWE in polar coordinates with the GTFCD [23]. Povstenko [16, 24] investigated the solution of TFDWE in polar and cylindrical coordinates with the time Caputo fractional derivative, and he found the solution in the integral form only. Here, we will obtain the closed form series solution of the TFDWE in polar coordinates with the GTFCD in terms of the Polygamma function.

The rest of this paper is organized as follows: A statement of the problem is illustrated in Section 2. Some basic definitions of the generalized fractional derivatives are given in Section 3. The closed form series solution of the problem with different cases is investigated in Section 4. Finally, we end the paper with the conclusion section.

2. Statement of the problem

TFDWE could be expressed in either Cartesian, cylindrical, or spherical coordinates. The choice of coordinates depends mainly on the geometry of the domain with which we are dealing. As simple rules, choose the coordinate system which makes the boundary conditions easy to apply, as introduced in [25], which solves the flow in pipes, axial, radial and torsional using polar cylindrical coordinates [25].

Consider the TFDWE with a source term in polar coordinates that is defined as follows [26]:

$${}_{a}^{c}D_{g}^{\alpha} T(r,t) = A \left(T_{rr}(r,t) + \frac{1}{r} T_{r}(r,t)\right) + Q(r,t), \;\; 0 < r \leqslant \infty \;\;\; a < t \leqslant \infty. \tag{2}$$

The initial conditions are given by

$$T(r,a) = F_1(r), \;\; 0 \leq \alpha \leq 2, \tag{3}$$

$$T_t(r,a) = F_2(r), \;\; 1 \leq \alpha \leq 2, \tag{4}$$

where T is the diffusion-wave variable, r is the radial coordinate, A is a constant, t is the time, $Q(r,t)$ is the source term and ${}_{a}^{c}D_{g}^{\alpha} T(r,t)$ is the generalized Caputo derivative of T with respect to the function g of order α, which is given by [23, 27, 28]

$$ {}_{a}^{C}D_{g}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \big(g(t) - g(a)\big)^{n-\alpha-1} f^{(n)}(u)\, g'(u)\, du, \tag{5} $$

Where $n-1 < \alpha < n$ and $\Gamma(\alpha)$ is the gamma function. The solution of Eq. (2) will be obtained using the generalized Laplace transform [23, 29] and Hankel transform [30] in the following three cases:

Case 1: At $Q(r,t) = 0,\ F_1(r) = P\frac{\delta(r)}{r},\ F_2(r) = 0,$ (6)

Case 2: At $Q(r,t) = 0,\ F_1(r) = 0,\ F_2(r) = P\frac{\delta(r)}{r},$ (7)

Case 3: At $Q(r,t) = \delta((g(t) - g(a))\frac{\delta(r)}{r},\ F_1(r) = 0,\ F_2(r) = 0,$ (8)

where P is a constant, $\delta(r)\ and\ \delta(t)$ are Dirac delta functions.

3. Some basic definitions

In this section, we introduce some basic definitions.

Definition 1. [31] The PolyGamma function $\psi(\boldsymbol{z})$ is given by

$$ \psi(z) = \Gamma'(z)\, \Gamma(z)^{-1}. \tag{9} $$

Definition 2. [30] The zeroth-order Hankel transform of the function $f(r)$ is given by

$$ \mathcal{H}_0\{f(r)\} = \tilde{f}_0(K) = \int_{0}^{\infty} r\, J_0(rK) f(r)\, dr, \tag{10} $$

where $J_0(z)$ is the zeroth order first kind Bessel function.

Definition 3. [30] The inverse zeroth-order Hankel transform of the function $\tilde{f}_0(K)$ is given by

$$ \mathcal{H}_0{}^{-1}\{\tilde{f}_0(K)\} = f(r) = \int_{0}^{\infty} K\, J_0(rK) \tilde{f}_0(K)\, dK. \tag{11} $$

Lemma 1. [30] The following identities holds true

$$ 1 - \mathcal{H}_0 \left\{ \frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr}) \right\} = -\, K^2\, \tilde{f}_0(K). \tag{12} $$

$$ 2 - \mathcal{H}_0 \left\{ \frac{\delta(r)}{r} \right\} = 1. \tag{13} $$

Lemma 2. [23] The generalized Laplace transform of the operator (5) is

$$L_g\{{}_a^cD_g^\alpha f(t)\} = s^\alpha\left(L_g\{f(t)\} - \sum_{k=0}^{n-1} s^{-k-1} f^{(k)}(a)\right). \tag{14}$$

Lemma 3. [32] The inverse generalized Laplace transform of the modified Bessel function of the second kind $\mathcal{K}_0$ is given by

$$\begin{gathered} {L_g}^{-1}\left\{\frac{1}{s^\mu}\,\mathcal{K}_0\left(\frac{a}{s^{\frac{m}{2}}}\right)\right\} = \sum_{k=0}^{\infty} \frac{a^{2k}((g(t)-g(a))^{mk+\mu-1}}{2*4^k\,(k!)^2\,\Gamma(\mu+mk)}\cdot \\ \cdot\left(2\psi(k+1) + m\psi(\mu+mk) - \ln\left(\frac{a^2((g(t)-g(a))^m}{4}\right)\right), \end{gathered} \tag{15}$$

where a, μ and m are constants.
Proof. See appendix A.

4. Closed form series solution of Eq. (2)

Applying zeroth-order Hankel transform (10) to Eq. (2), we get

$${}_a^cD_g^\alpha T(K,t) = A\mathcal{H}_0\{(T_{rr}(r,t) + \frac{1}{r}T_r(r,t))\} + Q(K,t), \tag{16}$$

Substituting Eq. (12) into Eq. (16), we get

$${}_a^cD_g^\alpha T(K,t) = -AK^2T(K,t)) + Q(K,t). \tag{17}$$

Apply the generalized Laplace transform (14) to Eq. (17) to get

$$\begin{gathered} s^\alpha L_g\big(T(K,t)\big) - s^{\alpha-1}T(K,a) - s^{\alpha-2}T_t(K,a) = -A\,k^2\,L_g\big(T(K,t)\big) + \\ + L_g(Q(K,t)). \end{gathered} \tag{18}$$

Applying conditions $T(K,a) = F_1(K)$ and $T_t(K,a) = F_2(K)$ in Eq. (18), we obtain

$$L_g(T(K,t)) = \frac{s^{\alpha-1}}{s^\alpha + AK^2}F_1(K) + \frac{s^{\alpha-2}}{s^\alpha + AK^2}F_2(K) + \frac{L_g(Q(K,t))}{s^\alpha + AK^2}. \tag{19}$$

4.1. Closed form series solution of Eq. (2) for case 1

Applying zeroth-order Hankel transform (10) to Eq. (6) and using Eq. (13), we obtain

$$Q(K,t) = 0,\ F_1(K) = P,\ F_2(K) = 0. \tag{20}$$

Substituting Eq. (20) into Eq. (19), we obtain

$$L_g\big(T(K,t)\big) = \frac{s^{\alpha-1}}{s^\alpha + AK^2}\, P. \tag{21}$$

Applying the inverse zeroth-order Hankel transform (11) to Eq. (21), we get

$$L_g(T(K,t) = P\int_0^\infty KJ_0(rK)\frac{s^{\alpha-1}}{s^\alpha + AK^2}dK = \tfrac{P}{A}s^{\alpha-1}\,\mathcal{K}_0\left(\frac{r}{\sqrt{As^{-\alpha}}}\right). \tag{22}$$

Apply the inverse generalized Laplace transform (15) to Eq. (21) to obtain

$$\begin{aligned} T(r,t) = \sum_{k=0}^{\infty}\frac{r^{2k}\,P\,\big(g(t)-g(a)\big)^{-\alpha-\alpha k}}{2 * A^{k+1} * 4^k\,(k!)^2\,\Gamma(1-\alpha-\alpha k)}\cdot \\ \cdot\left(2\psi(k+1) - \alpha\psi(1-\alpha-\alpha k) - \ln\left(\frac{r^2\big(g(t)-g(a)\big)^{-\alpha}}{4A}\right)\right). \end{aligned} \tag{23}$$

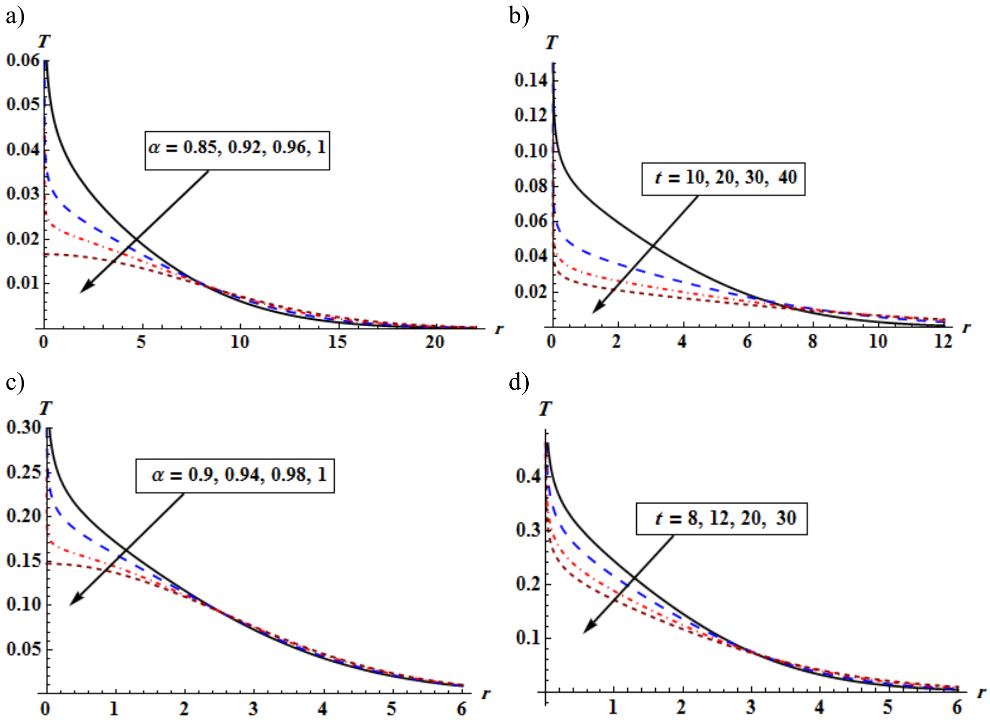

Fig. 1. The diffusion profile (23) when: a) $t = 30$, $g(t) = t$, $P = 1$, $A = 1$, $a = 0$, at different values of α; b) $\alpha = 0.9$, $g(t) = t$, $P = 1$, $A = 1$, $a = 0$, at different values of time; c) $t = 30$, $g(t) = \ln(t)$, $P = 1$, $A = 1$, $a = 1$, at different values of α; d) $\alpha = 0.9$, $g(t) = \ln(t)$, $P = 1$, $A = 1$, $a = 1$, at different values of time

4.2. Closed form series solution of Eq. (2) for case 2

Applying zeroth-order Hankel transform (10) to Eq. (7) and using Eq. (13), we obtain

$$F_1(K) = 0,\ F_2(K) = P \text{ and } Q(K,t) = 0, \tag{24}$$

Substituting Eq. (24) into Eq. (19), we obtain

$$L_g(T(K,t)) = \frac{s^{\alpha-2}}{s^{\alpha} + AK^2} P. \tag{25}$$

Applying the inverse zeroth-order Hankel transform (11) to Eq. (25), we get

$$L_g(T(K,t) = P \int_0^{\infty} K\, J_0(rK) \frac{s^{\alpha-2}}{s^{\alpha} + AK^2} dK = \frac{P}{A} s^{\alpha-2}\, \mathcal{K}_0\left(\frac{r}{\sqrt{As^{-\alpha}}}\right). \tag{26}$$

Apply the inverse generalized Laplace transform (15) to Eq. (26) to obtain

$$\begin{aligned} T(r,t) = \sum_{k=0}^{\infty} \frac{r^{2k}\, P\, \big(g(t) - g(a)\big)^{1-\alpha-\alpha k}}{2 * A^{k+1} * 4^k (k!)^2\, \Gamma(2-\alpha-\alpha k)} \cdot \\ \cdot \left(2\psi(k+1) - \alpha\psi(2-\alpha-\alpha k) - \ln\left(\frac{r^2 \big(g(t) - g(a)\big)^{-\alpha}}{4A}\right)\right). \end{aligned} \tag{27}$$

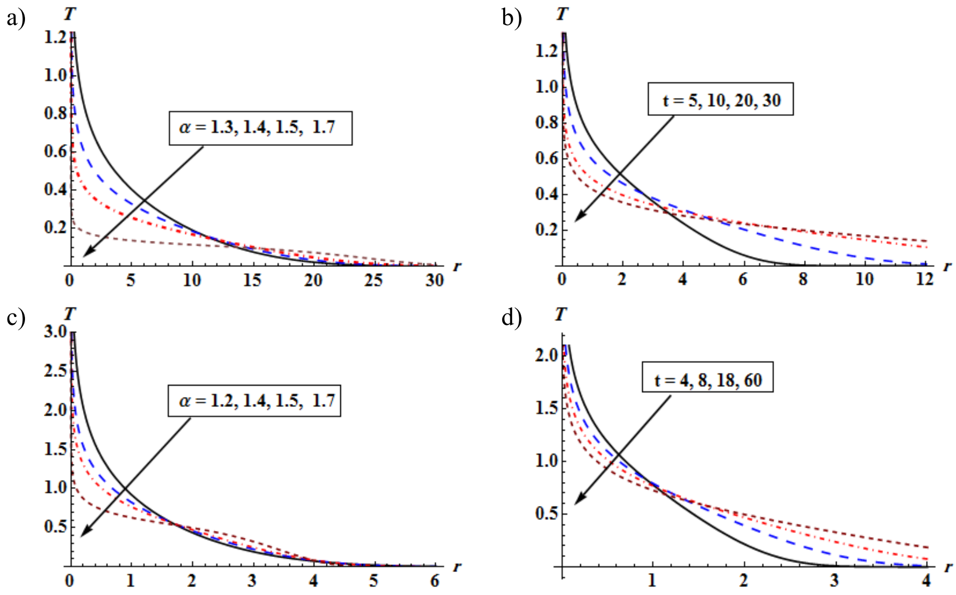

Fig. 2. The wave profile (27) when: a) $t = 30$, $g(t) = t$, $P = 1$, $A = 1$, $a = 0$, at different values of α; b) $\alpha = 1.5$, $g(t) = t$, $P = 1$, $A = 1$, $a = 0$, at different values of time; c) $t = 20$, $g(t) = \ln(t)$, $P = 1$, $A = 1$, $a = 0$, at different values of α; d) $\alpha = 1.5$, $g(t) = \ln(t)$, $P = 1$, $A = 1$, $a = 0$, at different values of time

4.3. Closed form series solution of Eq. (2) for case 3

Applying the zeroth-order Hankel transform (10) to Eq. (8) and using Eq. (13), we get

$$F_1(K) = 0. \; F_2(K) = 0 \; \text{ and } \; Q(K,t) = \delta((g(t) - g(a)), \tag{28}$$

Substituting Eq. (28) into Eq. (19), we obtain

$$L_g(T(K,t)) = \frac{1}{s^\alpha + AK^2}. \tag{29}$$

Applying the inverse zeroth-order Hankel transform (11) to Eq. (29), we get

$$L_g(T(K,t) = P\int_0^\infty KJ_0(rK)\frac{1}{s^\alpha + AK^2}dk = \frac{1}{A}\mathcal{K}_0\left(\frac{r}{\sqrt{As^{-\alpha}}}\right). \tag{30}$$

Apply the inverse generalized Laplace transform (15) to Eq. (30) to obtain

$$T(r,t) = \sum_{k=0}^{\infty}\frac{r^{2k}\left(g(t)-g(a)\right)^{-\alpha k-1}}{2A^{k+1}\,4^k\,(k!)^2\,\Gamma(-\alpha k)}\cdot \cdot\left(2\psi(k+1) - \alpha\psi(-\alpha k) - \ln\left(\frac{r^2\left(g(t)-g(a)\right)^{-\alpha}}{4A}\right)\right). \tag{31}$$

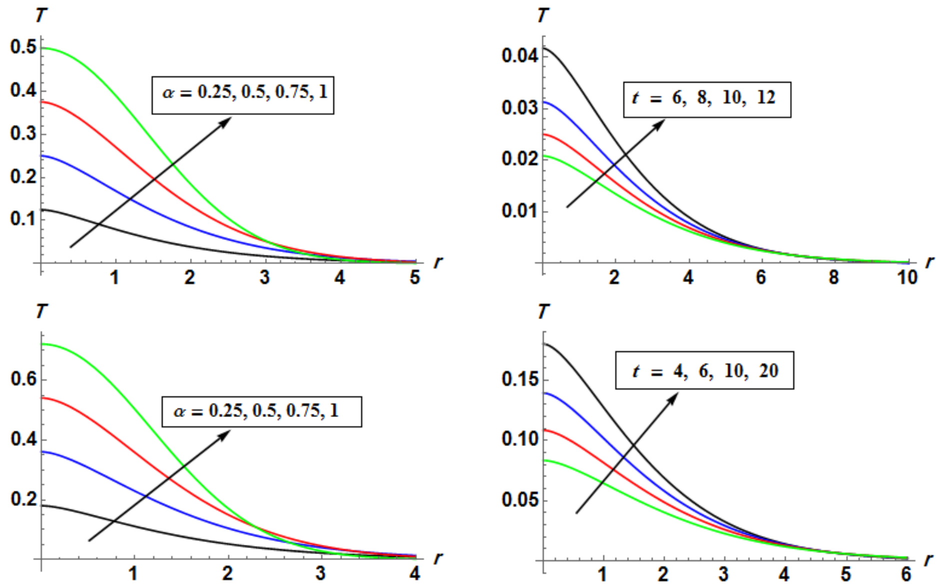

Fig. 3. The diffusion profile (31) when: a) $t = 1$, $g(t) = t$, $P = 1$, $A = 1$, $a = 0$, at different values of α; b) $\alpha = 0.5$, $g(t) = t$, $P = 1$, $A = 1$, $a = 0$, at different values of time; c) $t = 2$, $g(t) = \ln(t)$, $P = 1$, $A = 1$, $a = 0$, at different values of α; d) $\alpha = 0.5$, $g(t) = \ln(t)$, $P = 1$, $A = 1$, $a = 0$, at different values of time

5. Discussion and conclusions

In the case of $g(t) = t$, Figures 1a, 1b, 2a, 2b, 3a and 3b show the effect of the parameter α and time t on the diffusion-wave profile in the case of the Caputo fractional derivative. In the case of $(t) = \ln(t)$, Figures 1c, 1d, 2c, 2d, 3c, and 3d show the effect of the parameter α and time on the diffusion-wave profile in the case of fractional Hadamard fractional derivative. In general, from Figures 1-3, it can be observed that the diffusion-wave profile decays with increasing r. Furthermore, it is observable from Figures 1, and 2 that at small values of r, the diffusion-wave profile decreases with an increase in the value of α and t, whereas at large values of r, the diffusion-wave profile increases with an increase in the value of α and time t. Also, we can realize that from Figure 3, the diffusion profile increases with an increase in the value of α and time t. So, we can conclude that the generalized Laplace and Hankel transforms are utilized as effective tools in solving TFDWE with the GTFCD. New solutions of the linear TFDWE in polar coordinates with a source term are obtained for some different cases of the initial conditions. The results are illustrated graphically for some different cases.

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Appendix A

Theorem [16], Let, g: $[a,\infty) \to R$ be real valued functions such that $g(t)$ is continuous and $g'(t) > 0$ on $[0,\infty)$ and such that generalized laplace transform of f exists. Then

$$L_g\{f(t)\}(s) = L\left\{f\left(g^{-1}(t+g(a))\right)\right\}(s), \tag{A1}$$

Where $L\{f\}$ is the usual Laplace transform of f. The inverse Laplace transform of the modified Bessel function of the second kind $\mathcal{K}_0$ [21] is

$$L^{-1}\left\{\frac{1}{s^{\mu}}\,\mathcal{K}_0\left(\frac{a}{s^{\frac{m}{2}}}\right)\right\} = \sum_{k=0}^{\infty}\frac{a^{2k}\,t^{mk+\mu-1}}{2*4^k\,(k!)^2\,\Gamma(\mu+mk)} \left(2\psi(k+1) + m\psi(\mu+mk) - \ln\left(\frac{a^2t^m}{4}\right)\right). \tag{A2}$$

So, the inverse generalized Laplace transform of modified Bessel function second kind $\mathcal{K}_0$ is

$$L_g{}^{-1}\left\{\frac{1}{s^{\mu}}\,\mathcal{K}_0\left(\frac{a}{s^{\frac{m}{2}}}\right)\right\} = \sum_{k=0}^{\infty}\frac{a^{2k}\,((g(t)-g(a))^{mk+\mu-1}}{2*4^k\,(k!)^2\,\Gamma(\mu+mk)} \left(2\psi(k+1) + m\psi(\mu+mk) - \ln\left(\frac{a^2((g(t)-g(a))^m}{4}\right)\right). \tag{A3}$$