JAM CM Journal of Applied Mathematics and Computational Mechanics 2023, 22(4), 5-17
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2023.4.01 e-ISSN 2353-0588

SEMI-ANALYTICAL SCHEME WITH ITS STABILITY ANALYSIS FOR SOLVING THE FRACTIONAL-ORDER PREDATOR-PREY EQUATIONS BY USING LAPLACE-VIM

Mohamed Adel[1], Nasser H. Sweilam[2], Mohamed M. Khader[3,4]

[1] *Department of Mathematics, Faculty of Science, Islamic University of Madinah*
Medina, KSA
[2] *Department of Mathematics, Faculty of Science, Cairo University*
Giza, Egypt
[3] *Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU)*
Riyadh, KSA
[4] *Department of Mathematics, Faculty of Science, Benha University*
Benha, Egypt
adel@sci.cu.edu.eg, nsweilam@sci.cu.edu.eg, mmkhader@imamu.edu.sa

Received: 22 July 2023; Accepted: 29 October 2023

Abstract. The article's goal is to implement a semi-analytical technique named, the Laplace variational iteration method (LVIM), which is the combination of VIM and Laplace transform method. Although both the Laplace transform method and VIM cannot be applied to some nonlinear fractional differential equations (FDEs) individually, this combination will give a fast-convergent solution to the problem under study. The proposed scheme is used to numerically solve a biodynamic system called the Lotka-Volterra system, i.e. Predator-Prey Equations (PPEs). The system of FDEs can be used to represent this scenario, as well as the Caputo-Fabrizio fractional derivative will be used throughout the study. By assessing the residual error function, we can confirm that the given procedure is effective and accurate. The outcomes demonstrate that the technique used is an effective tool for simulating such models.

MSC 2010: *41A10, 65N12, 65N35*
Keywords: *fractional predator-prey equations, Caputo-Fabrizio fractional derivative operator, Laplace transform, VIM, fixed point*

1. Introduction

Over the past three decades, numerous authors have maintained interest in fractional calculus [1–3]. To fulfill the demand to model real-world problems in various domains like mechanics, biology, and engineering [4, 5], several academics have discovered that developing new fractional derivatives with various singular or non-singular kernels is crucial. The majority of FDEs lack an exact solution, so numerical and approximate techniques must be used [6].

Recently, Caputo and Fabrizio (CF) have proposed a new operator by replacing the singular kernel in the classical Caputo derivative with the regular kernel [7]. The CF operator uses the exponential kernel, which is a non-singular kernel. It not only has two different representations for temporal and spatial variables, but the full effect of the memory can also be portrayed [8]. This new operator has been successfully applied in many problems, such as the space time-fractional diffusion equation [9], fractional glioblastoma multiforme [10], and others [11].

In the present study, we use two techniques: The VIM [12] and the Laplace transform method [13,14]. Then the combined technique, the Laplace variational iteration method (LVIM) [15], is used to investigate the approximate solutions for the fractional PPEs. The main advantage of this technique is the faster convergence compared with the existing one from related literature. In addition, LVIM is implemented to solve some problems, such as the nonlinear fractional order Burger's equation [16].

2. Preliminaries and notations

2.1. Some definitions of fractional calculus and Laplace transforms

Definition 1. The Caputo-Fabrizio fractional derivative ${}^{cf}D^{\nu}$ of order $0 \leq \nu \leq 1$ of a function $\varphi(t) \in H^1(0,a)$, is defined in the following form:

$$ {}^{cf}D^{\nu}\varphi(t) = \frac{1}{1-\nu}\int_0^t \varphi'(\tau)\, e^{\frac{-\nu(t-\tau)}{1-\nu}}\, d\tau, \qquad t \geq 0. $$

Definition 2. Let $\phi(t)$ be a function, then its Laplace transform is defined as:

$$ Ł\{\phi(t)\} = \Phi(s) = \int_0^{\infty} e^{-st}\,\phi(t)dt, \tag{1} $$

and the Laplace transform of $\phi(t)$ in the Caputo-Fabrizio sense is given by [8]:

$$ Ł\{{}^{cf}D^{\nu+m}\phi(t)\} = \frac{s^{m+1}Ł\{\phi(t)\} - s^m\phi(0) - s^{m-1}\phi'(0) - \ldots - \phi^{(m)}(0)}{s+\nu(1-s)}. \tag{2} $$

For $m = 0$, we have:

$$ Ł\{{}^{cf}D^{\nu}\phi(t)\} = \frac{sŁ\{\phi(t)\} - \phi(0)}{s+\nu(1-s)}. \tag{3} $$

2.2. Some stability concepts

Many stability concepts have been developed within the last few decades, among which are Lyapunov stability [17], exponential stability [18], and so on. Next, we use the Banach contraction principle to check the stability of the proposed LVIM. For this, let us recall some of the main definitions from the fixed point theory.

Definition 3. Let $(\Omega, |.|)$ be a metric space. A mapping $A : \Omega \to \Omega$ is said to be a contraction mapping, if for all $\varphi_1, \varphi_2 \in \Omega$ and a positive real constant $\gamma < 1$ we have:

$$|A\varphi_1 - A\varphi_2| \leq \gamma |\varphi_1 - \varphi_2|. \tag{4}$$

This means that any pair of points $\varphi_1, \varphi_2 \in \Omega$ have images closer than the points φ_1, φ_2 or, in other words, the ratio,

$$\frac{|A\varphi_1 - A\varphi_2|}{|\varphi_1 - \varphi_2|},$$

does not exceed a positive constant γ, which is less than one.

Furthermore, let us recall Picard's existence and uniqueness theorem for differential equations. So we consider the following first-order initial value problem:

$$\dot{u}(t) = A(t;u), \qquad u(t_0) = u_0, \tag{5}$$

with t_0 and u_0 two given real numbers. Let A be a continuous mapping on the rectangle

$$\bar{R} = \{(t;u) : \ |t - t_0| \leq a, \ |u - u_0| \leq b\}.$$

Thus, A is bounded on $\bar{R}$. Then, for all $(t;u) \in \bar{R}$, we can write $|A(t;u)| \leq c$.
Suppose that A satisfies the Lipschitz condition on $\bar{R}$ concerning its second argument. Then, there exists a Lipschitz constant γ such that, for all $(t;u), (t;v) \in \bar{R}$,

$$|A(t;u) - A(t;v)| \leq \gamma |u - v|. \tag{6}$$

In these conditions, the above initial value problem (5) has a unique solution in the interval $(t_0 - \sigma, t_0 + \sigma)$, where $\sigma < \{a, \frac{b}{c}, \frac{1}{\gamma}\}$.

2.3. The fractional predator-prey system

It is well-known that the predator and prey equations are a pair of nonlinear first-order differential equations that are used to describe the dynamics of some biological systems in which two species interact with each other, one being a predator and the other being a prey. Therefore, Samardzija & Greller [19] made an extension of this model and proposed the concept of predators and single prey for the Lotka-Volterra system. This system is what we are going to study here, and it is formulated in its fractional form as follows:

$$D^{\nu}\Phi(t) = \sigma_1 \Phi(t) - \sigma_2 \Phi(t)\Psi(t) + \sigma_3 \Phi^2(t) - \sigma_4 \Upsilon(t) \Phi^2(t), \tag{7}$$

$$D^{\nu}\Psi(t) = -\sigma_5\,\Psi(t) + \sigma_6\,\Phi(t)\Psi(t), \tag{8}$$

$$D^{\nu}\Upsilon(t) = -\sigma_7\,\Upsilon(t) + \sigma_4\,\Upsilon(t)\,\Phi^2(t), \tag{9}$$

$$\Phi(0) = c_1, \qquad \Psi(0) = c_2, \qquad \Upsilon(0) = c_3. \tag{10}$$

where c_1, c_2, and c_3 are constants. Here the predators are $\Phi(t)$ and $\Psi(t)$, and its prey is $\Upsilon(t)$, through the time t; σ_1, σ_2, σ_3, σ_4, σ_5, σ_6, and σ_7 are parameters that elucidate the interaction between the three species [19].

3. Basis of the Laplace-VIM

This section is focused on presenting the basis of the Laplace-VIM. For this, we consider the following FDE:

$$D^{\nu}u(t) + L(u(t)) + N(u(t)) = g(t), \qquad \text{subject to} \qquad u(0) = u_0, \tag{11}$$

where $L(u(t)), N(u(t))$, and $g(t)$ are linear, nonlinear and known functions, respectively. Additionally, we consider D^{ν} in the CF-sense.

Applying the VIM in Equation (11), we get:

$$u_{n+1}(t) = u_n(t) + \lambda\,[D^{\nu}u(t) + L(u(t)) + N(u(t)) - g(t)].$$

Additionally, applying the Laplace transform Ł, the variable t becomes a new one s, such that:

$$u_{n+1}(s) = u_n(s) + \lambda\,Ł[D^{\nu}u_n(t) + L(\tilde{u}_n(t)) + N(\tilde{u}_n(t)) - g(t)], \tag{12}$$

where $\tilde{u}(t)$ etc. are restricted values, which means $\delta\tilde{u}_n(t) = 0$. Using relation Equation (3), we get:

$$Ł\{D^{\nu}u_n(t)\} = \frac{s\,u_n(s) - u_n(0)}{s + \nu(1-s)}, \qquad Ł\{\delta\,D^{\nu}u_n(t)\} = \frac{s\,\delta\,u_n(s) - \delta\,u_n(0)}{s + \nu(1-s)},$$

where $\delta\,u_n(0) = 0$. Then, we obtain:

$$Ł\{\delta\,D^{\nu}u_n(t)\} = \frac{s\,\delta\,u_n(s)}{s + \nu(1-s)}.$$

From the optimization conditions,

$$\frac{\delta\,\tilde{u}_{n+1}(s)}{\delta\,\tilde{u}_n(s)} = 0, \qquad \delta\,\tilde{u}_n = 0,$$

we get:

$$1+\lambda\left[\frac{s\,\delta\,\tilde{u}_n(s)}{s+\nu(1-s)\delta\,\tilde{u}_n(s)}\right]=0.$$

The above equation gives $\lambda=-\dfrac{s+\nu(1-s)}{s}$.
Replacing Equation (12), we obtain:

$$u_{n+1}(s)=u_n(s)-\Big(\frac{s+\nu(1-s)}{s}\Big)\text{Ł}[D^{\nu}u_n(t)+L(u_n(t))+N(u_n(t))-g(t)].$$

Using the inverse Laplace transform, we get:

$$u_{n+1}(t)=u_n(t)-\text{Ł}^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big)\text{Ł}[D^{\nu}u_n+L(u_n)+N(u_n)-g(t)]\Big]. \tag{13}$$

Replacing $n=0,1,2,...$, we can get the following successive approximations, $u_1(t)$, $u_2(t),...$, which converge to exact solution, i.e.

$$u(t)=\lim_{n\to\infty}u_n(t).$$

4. Stability analysis of the LVIM scheme

Theorem 1. The Laplace-VIM scheme (13) is unconditionally stable.

PROOF Considering the above discussion in Section 2 and the scheme (13) to check the stability of LVIM as follows:

$$u_{n+1}(t)=A(u_{n+1})=u_n(t)-\text{Ł}^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big)\text{Ł}[D^{\nu}u_n(t)+L(u_n(t))+N(u_n(t))-g(t)]\Big].$$

Then we have:

$$\begin{aligned}|A(u_{n+1})-A(u_{m+1})|\leq\ |u_n-u_m|-\text{Ł}^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big)\text{Ł}\Big[D^{\nu}(u_n-u_m)\\ +L(u_n-u_m)+N(u_n-u_m)\Big]\Big],\end{aligned}$$

which implies,

$$\begin{aligned}\frac{|A(u_{n+1})-A(u_{m+1})|}{|u_n-u_m|}\leq 1-\text{Ł}^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big)\text{Ł}\Big[D^{\nu}(u_n-u_m)\\ +L(|u_n-u_m|)+N(|(u_n-u_m)|)\Big]\Big]=\gamma.\end{aligned}$$

Then, we obtain:

$$\frac{|A(u_{n+1})-A(u_{m+1})|}{|u_n-u_m|}\leq\gamma,$$

which means $|A(u_{n+1}) - A(u_{m+1})| \leq \gamma |u_n - u_m|$. Then, the proposed scheme (13) is unconditionally stable by Equation (6). ■

Remark 1. As a remark concerning LVIM on nonlinear FDEs is the existence of the approximations of the solution $u_1(t), u_2(t), u_3(t), ...$ From Section 3 it is easy to conclude that these approximations of the solution are similar to the iterations of a sequence of successive approximations which is convergent to a fixed point. Then, we get a strong connection between the proposed scheme and the fixed point theory. It would be very interesting to check in which conditions we get a fixed point for this type of approximation.

5. Implementation of Laplace VIM via PPEs

The principles of the LVIM and its applicability for various kinds of differential equations are given in [20]. In [21], it was shown that the VIM is also valid for fractional differential equations. In this section, by following the discussion presented in [21], we extend the application of the LVIM to solve the model of the fractional system described in the previous subsection 2.3.

According to the LVIM, we can construct the correction functional for the system (7)-(9) as follows:

$$\begin{aligned}\Phi_{m+1}(s) = \Phi_m(s) + \lambda_1 \text{Ł}\Big[& D^\nu \Phi_m(t) - \sigma_1 \Phi_m(t) + \sigma_2 \Phi_m(t)\Psi_m(t) \\ & - \sigma_3 \Phi_m^2(t) + \sigma_4 \Upsilon_m(t)\Phi_m^2(t)\Big],\end{aligned} \tag{14}$$

$$\Psi_{m+1}(s) = \Psi_m(s) + \lambda_2 \text{Ł}\left[D^\nu \Psi_m(t) + \sigma_5 \Psi_m(t) + \sigma_6 \Phi_m(t)\Psi_m(t)\right], \tag{15}$$

$$\Upsilon_{m+1}(s) = \Upsilon_m(s) + \lambda_3 \text{Ł}\left[D^\nu \Upsilon_m(t) + \sigma_7 \Upsilon_m(t) - \sigma_4 \Upsilon_m(t)\Phi_m^2(t)\right], \tag{16}$$

where λ_k, $(k = 1,2,3)$ are the general Lagrange multipliers, which can be identified optimally via variational theory [22]. The optimality conditions give the following results:

$$\frac{\delta \tilde{\Phi}_{m+1}(s)}{\delta \tilde{\Phi}_m(s)} = 0, \quad \frac{\delta \tilde{\Psi}_{m+1}(s)}{\delta \tilde{\Psi}_m(s)} = 0, \quad \frac{\delta \tilde{\Upsilon}_{m+1}(s)}{\delta \tilde{\Upsilon}_m(s)} = 0, \quad \delta\tilde{\Phi}_m = \delta\tilde{\Psi}_m = \delta\tilde{\Upsilon}_m = 0,$$

where $\tilde{\Phi}_m$, $\tilde{\Psi}_m$, and $\tilde{\Upsilon}_m$ are considered as restricted variations, and the above equations give:

$$\lambda_1 = \lambda_2 = \lambda_3 = -\frac{s + \nu(1-s)}{s}.$$

Replacing $\lambda_1, \lambda_2, \lambda_3$ into the functional equations (14)-(16) and using inverse Laplace transform to obtain the following iteration formula:

$$\begin{aligned}\Phi_{m+1}(t) = \Phi_m(t) - \text{Ł}^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big)\cdot \\ \Big(D^{\nu}\Phi_m - \sigma_1\Phi_m + \sigma_2\Phi_m\Psi_m - \sigma_3\Phi_m^2 + \sigma_4\Upsilon_m\Phi_m^2\Big)\Big],\end{aligned} \tag{17}$$

$$\Psi_{m+1}(t) = \Psi_m(t) - \text{Ł}^{-1}\left[\Big(\frac{s+\nu(1-s)}{s}\Big)\Big(D^{\nu}\Psi_m + \sigma_5\Psi_m + \sigma_6\Phi_m\Psi_m\Big)\right], \tag{18}$$

$$\Upsilon_{m+1}(t) = \Upsilon_m(t) - \text{Ł}^{-1}\left[\Big(\frac{s+\nu(1-s)}{s}\Big)\Big(D^{\nu}\Upsilon_m + \sigma_7\Upsilon_m - \sigma_4\Upsilon_m\Phi_m^2\Big)\right]. \tag{19}$$

The initial approximations $\Phi_0(t)$, $\Psi_0(t)$, and $\Upsilon_0(t)$ can be freely chosen if they satisfy the initial conditions of the problem. Finally, we approximate the solutions $\Phi(t)$, $\Psi(t)$, and $\Upsilon(t)$ by the m-th terms $\Phi_m(t)$, $\Psi_m(t)$, and $\Upsilon_m(t)$, respectively as follows:

$$\Phi(t) = \lim_{m\to\infty}\Phi_m(t), \qquad \Psi(t) = \lim_{m\to\infty}\Psi_m(t), \qquad \Upsilon(t) = \lim_{m\to\infty}\Upsilon_m(t). \tag{20}$$

6. Numerical applications: Approximate solution for PPEs

We are going to verify the accuracy and quality of the given scheme by presenting a numerical simulation on a test example, where we address the system (7)-(9) with different values of ν, m; with $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$, $\sigma_5 = 2$, $\sigma_6 = 3$, $\sigma_7 = 2.7$ and initial conditions $\Phi_0 = 1$, $\Psi_0 = 1.4$, $\Upsilon_0 = 1$. The obtained numerical results for the studied model by applying the proposed technique are introduced through Figures 1-3.

All codes were written and debugged by Mathematica 11 on a Dell Inspiron 15 (3593) Workstation, Processor: 11th Gen Intel(R) Core(TM) i7-1165G7 and 2.80 GHz 1.69 GHz, 32 GB Ram DDR3, and 1 TB of storage.

By using the variational iteration formula (17)-(19), if we start with the initial approximations $\Phi_0(t) = 1$, $\Psi_0(t) = 1.4$, and $\Upsilon_0(t) = 1$, we can directly obtain some of the other components as follows:

$$\Phi_1(t) = 1 - \frac{0.4t^{\nu}}{\Gamma(1+\nu)}, \qquad \Psi_1(t) = 1.4 + \frac{1.417t^{\nu}}{\Gamma(1+\nu)}, \qquad \Upsilon_1(t) = 1 - \frac{1.7t^{\nu}}{\Gamma(1+\nu)},$$

$$\Phi_2(t) = 1 - \frac{0.4t^{\nu}}{\Gamma(1+\nu)} + \frac{0.46t^{2\nu}}{\Gamma(1+2\nu)} - \frac{1.553t^{3\nu}}{\Gamma(1+3\nu)} + \frac{1.515t^{4\nu}}{\Gamma(1+4\nu)}, \ldots,$$

$$\Psi_2(t) = 1.4 + \frac{1.417t^{\nu}}{\Gamma(1+\nu)} - \frac{0.148t^{2\nu}}{\Gamma(1+2\nu)} - \frac{0.608t^{3\nu}}{\Gamma(1+3\nu)}, \ldots,$$

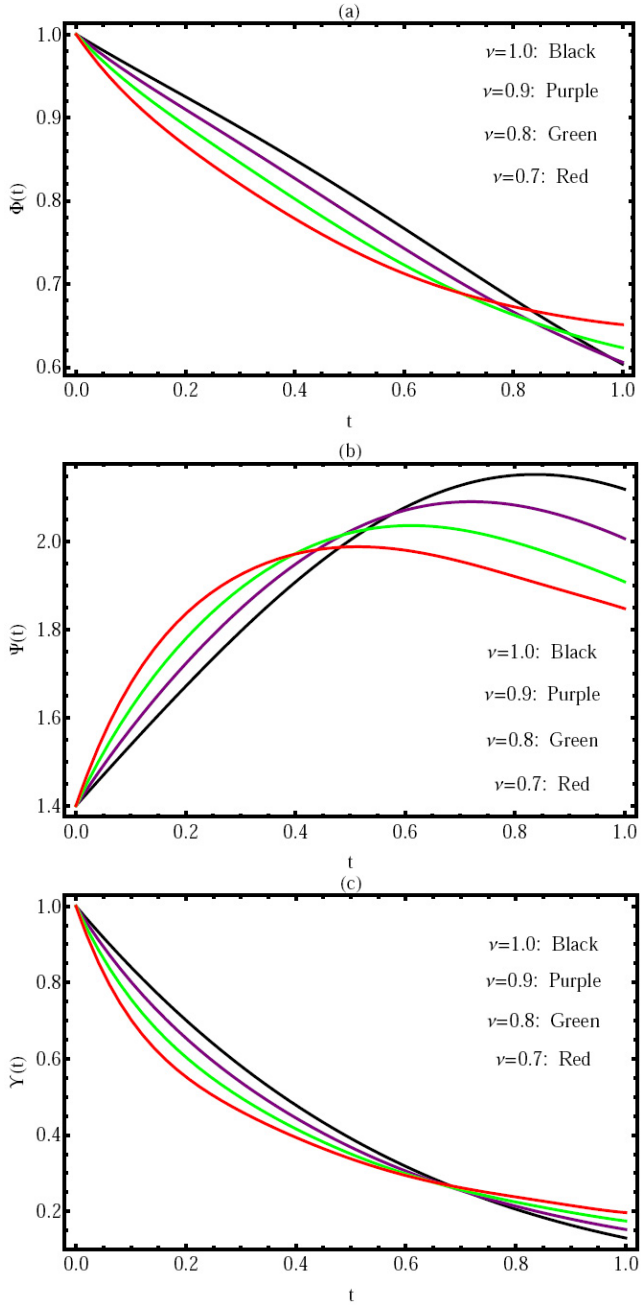

Fig. 1. The approximate solution $\Phi(t)$, $\Psi(t)$, $\Upsilon(t)$ against distinct values of ν

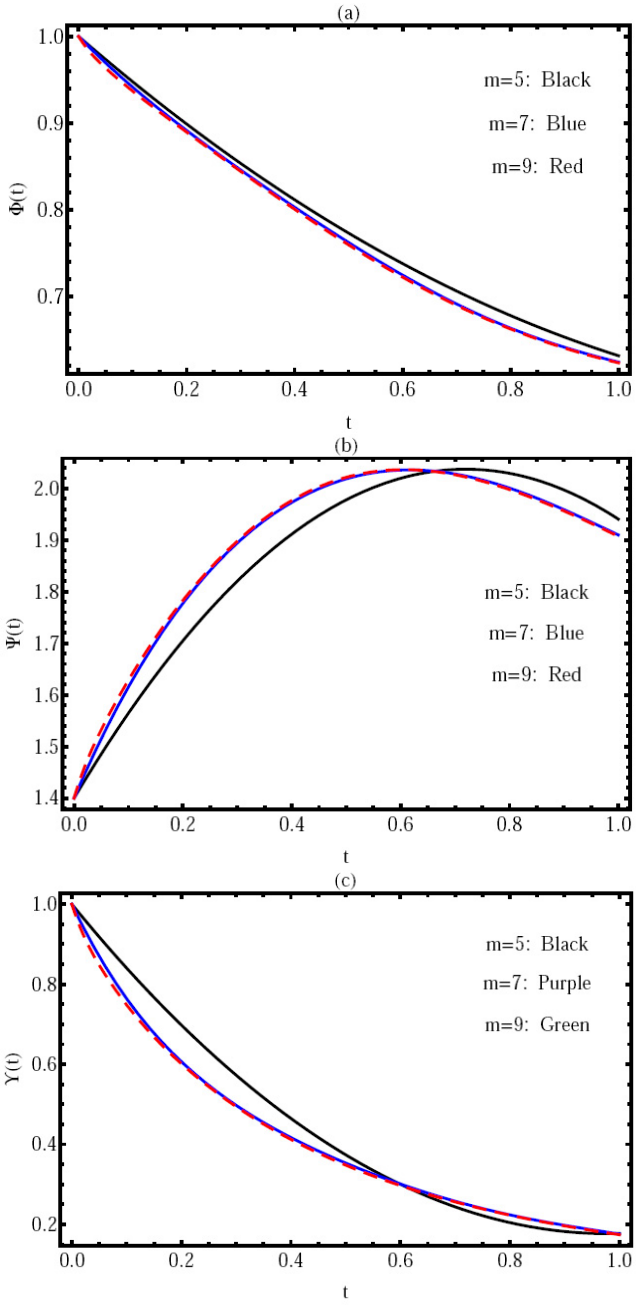

Fig. 2. The approximate solution $\Phi(t)$, $\Psi(t)$, $\Upsilon(t)$ against distinct values of m

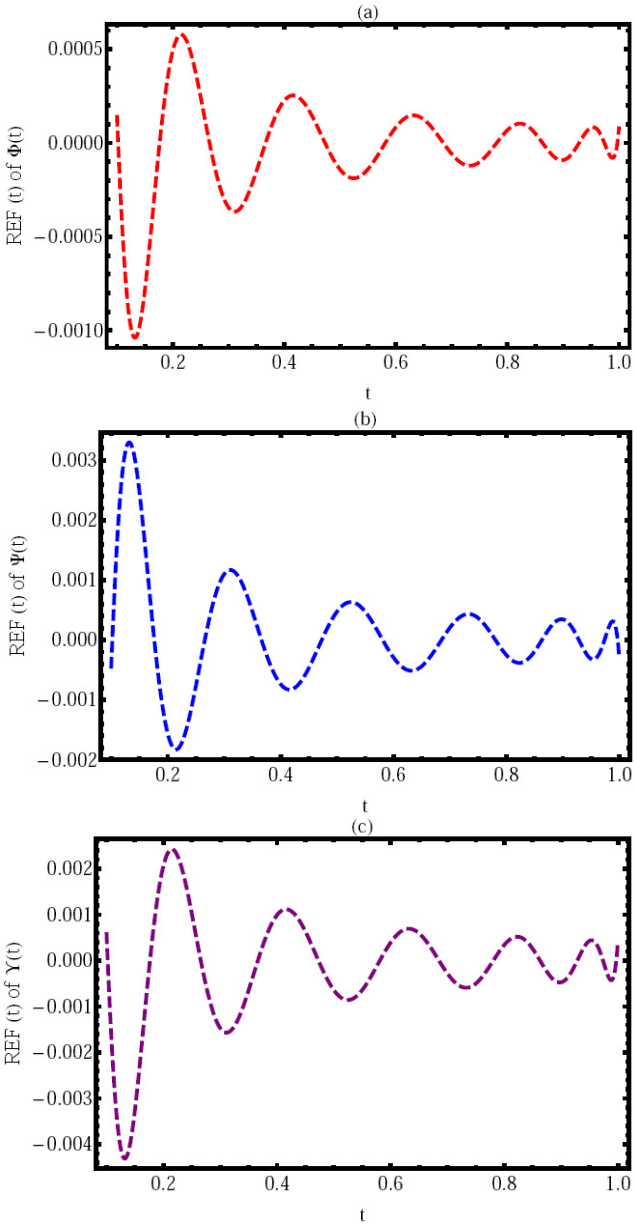

Fig. 3. The REF of the solutions $\Phi(t), \Psi(t), \Upsilon(t)$

$$\Upsilon_2(t) = 1 - \frac{1.7t^{\nu}}{\Gamma(1+\nu)} + \frac{2.1t^{2\nu}}{\Gamma(1+2\nu)} + \frac{2.95t^{3\nu}}{\Gamma(1+3\nu)} - \frac{1.515t^{4\nu}}{\Gamma(1+4\nu)}, ...,$$

and so on; in the same way the rest of the components of the iteration formula can be obtained. Here, in our computation, we approximated the solution $\Phi(t)$, $\Psi(t)$, and $\Upsilon(t)$, by $\Phi(t) \cong \Phi_7(t)$, $\Psi(t) \cong \Psi_7(t)$, and $\Upsilon(t) \cong \Upsilon_7(t)$, respectively.

Where the behavior of the approximate solution via distinct values of $\nu = 1.0, 0.9, 0.8, 0.7$, with $m = 5$ is given in Figure 1, but in Figure 2, we present the behavior of the approximate solution via distinct values of $m = 5, 7, 9$, with $\nu = 0.95$. Figure 3 is potted represent the residual error function (REF) [23] of the approximate solution at $\nu = 0.96$ with $m = 6$. Through these results, we note that the behavior of the numerical solution resulting from the application of the proposed method depends on the values of ν and m, and this confirms that the proposed method is suitable for solving the proposed model in its fractional form.

7. Conclusions

The Laplace VIM is a powerful method and a semi-analytic scheme able to handle linear/nonlinear fractional differential equations. The method has been applied to the fractional predator-prey equations to find their approximate solutions. The proposed scheme converges faster than the existing ones in the literature and is a more reliable technique. The results show that the applied method is suitable and inexpensive for obtaining the approximate solutions for the proposed two models. Also, we can control and decrease the absolute error by adding new terms from the components of the solution. In addition, the approximate solutions with different values of the fractional-order ν, the orde of approximation m, and the residual error function are computed to illustrate the validity of the proposed technique. We also put in evidence a relationship between the approximations of the solution and the iterations of the sequence of successive approximations used in the proof of the existence of a fixed point. The main advantage is that the proposed scheme streamlines the computational processes and can be used efficiently for nonlinear dynamical systems analysis based on software such as Mathematica. Finally, as a generalization of the current work, we intend to use the same model in future research but with different approximation methods, or other fractional derivatives, as well as the optimal control.

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