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CERTAIN CONVERGENCE RESULTS FOR HOMOGENEOUS SINGULAR YOUNG MEASURES

Piotr Puchała

Department of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
piotr.puchala@pcz.pl, p.st.puchala@gmail.com

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Abstract. We consider purely singular homogeneous Young measures associated with elements of sequences of piecewise constant functions and with limits of such sequences. We first consider a case when the limit of a such sequence is piecewise constant. The next point involves the sequences of bounded oscillating functions, divergent in the strong topology in L^∞, but weakly* convergent to a homogeneous Young measure. We also present an example of a fast oscillating sequence, illustrating the result. In the presented results, generalizing to some extent known examples, we try to avoid advanced methods of functional analysis that are usually used when solving problems of this type.

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1. Introduction

Young measures were discovered in the context of the variational problems concerning minimization of integral functionals that are bounded from below. The first work proving the existence of the objects referred to today as 'Young measures' and investigating their basic properties is [1]. O. Bolza and L.C. Young himself presented examples of bounded integral functionals that did not attain their infima. One of the most popular methods of minimization functionals in the calculus of variation is the so called direct method. It is based on analysis of the minimizing sequences of the considered functionals. The minimizing sequences in the examples by Bolza and Young are of a highly oscillatory nature. They are bounded but divergent in the suitable for the problem normed space (usually a Sobolev one). They are convergent in a weak (or weak*) topology. However, investigating their limits requires an enlargement of the space of admissible functions from the 'usual' space of functions with values in, say, $\mathbb{R}^l$, to the space of scalar-valued measures defined on the Borel σ-algebra of subsets of $\mathbb{R}^l$. These 'generalized limits', in a way, summarize the spatial oscillatory properties of the minimizing sequences. Unfortunately, the calculation of a generalized limit, i.e. a Young measure associated with a minimizing sequence, is usually very difficult.

Young measures appear in the mathematical analysis of certain engineering problems, for example in the investigation of the infima of the energy functionals of certain shape-memory alloys, like Ni-Ti or Cu-Al-Ni. Since the infima of the energy functionals in this case are not attained, the respective minimizing sequences are rapidly oscillating. These oscillations reveal a phenomenon called a *microstructure*. As it is observed in [2], "(...) a microstructure is any structure on a scale between the macroscopic scale (on which we usually make observations) and the atomic scale. Such structures are abundant in nature: the fine hierarchical structures in leaf and many other biological materials, the complex arrangements of fissures, cracks, (...), man-made layered or fibre-reinforced materials and fine phase mixtures in solid-solid phase transformations, to quote but a few examples".

The techniques involving the Young measures apparatus have turned out to be useful in investigating problems in differential equations with oscillatory phenomena. The considered problems are often of a physical or engineering origin. With this respect, the reader may have a look at [3–12] (and the references therein), from the broad spectrum of articles on this subject.

In this article, we look at a Young measure as a value of a weakly*-measurable mapping defined on a domain of definition of considered functions. We use the fact, that with any bounded, Borel, $\mathbb{R}^l$-valued function there exists a Young measure associated with it. This approach makes calculating generalized limits of sequences of rapidly oscillating functions in many practically significant cases easier. We focus on the purely singular case, when both the elements of function sequence and its limit are discrete.

The structure of the article is as follows. In the next section, we recall facts concerning Young measures together with some notions from functional analysis that are used in the article. The third part of the article, containing main results, is divided into two parts. In the first one we consider sequences of simple functions pointwise convergent to some simple function. The second part is devoted to those sequences of simple functions that are only weakly* convergent to generalized limits being respective Young measures. The result is illustrated by an example generalizing usual examples of purely discrete Young measures associated with sequences of rapidly oscillating functions. Finally, the Conclusions section closes the main body of the article.

2. Some necessary facts about Young measures

We begin with setting notation and formulating basic facts. For more detailed information and bibliographical suggestions for further reading on Young measures, we refer the reader to [13–16].

Let Ω be an open subset of $\mathbb{R}^d$, such that $\mu(\Omega) = M > 0$, where μ is the Lebesgue measure on Ω.

We first recall the notions of basic types of convergences in Banach spaces and the notion of weak*-measurability.

Definition 1 Let Z be a Banach space, Z^* – the conjugate of Z (i.e. the space of all continuous linear functionals on Z). Let (z_n) be a sequence in Z and (f_n) – a sequence in Z^*.

(a) we say that (z_n) is weakly convergent to $z \in Z$, if for all $f \in Z^*$ there holds $\lim_{n\to\infty} f(z_n) = f(z)$;

(b) we say that (f_n) is weakly* convergent to $f \in Z^*$, if for all $z \in Z$ there holds $\lim_{n\to\infty} f_n(z) = f(z)$;

(c) denote by $\langle \cdot,\cdot \rangle$ a dual pair that is a real valued mapping defined on $Z^* \times Z$, linear in each variable separately. We say that a mapping

$$g\colon \Omega \to Z^*$$

is *weakly*-measurable*, if for any $z \in Z$ the function

$$x \mapsto \langle g(x), z\rangle$$

is measurable. □

We will use the following notation:

- the letter K denotes a nonempty compact subset of $\mathbb{R}^l$ while $\mathscr{U}$ – the set of all Borel measurable functions on Ω with values in K;
- $rca(K)$ – the space of regular, countably additive scalar measures on K, equipped with the norm $\|m\|_{rca(K)} := |m|(K)$, where $|\cdot|$ stands in this case for the total variation of the measure m. By definition, $|m|(K) = \sup \sum_i |m(K_i)|$, where the supremum is taken over all partitions of the set K. With this norm, $rca(K)$ is a Banach space;
- $rca^1(K)$ – the subset of $rca(K)$ with elements being probability measures on K;
- $L^\infty_{w^*}(\Omega, \mathrm{rca}(K))$ – the set of the weakly* measurable mappings

$$\nu\colon \Omega \ni x \to \nu(x) \in rca(K),$$

assigning to the points from the domain of definition of $u \in \mathscr{U}$ the measures on the range of u and such that

$$\operatorname{ess\,sup}\big\{ \|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega \big\} < +\infty.$$

The Definition 1 applied to the case described by the above settings together with the Riesz Representation Theorem yield that ν is a weakly* measurable mapping if for any $\beta \in C(K)$ the function

$$x \mapsto \int_K \beta(k)(\nu(x))(dk) = \langle \nu(x), \beta\rangle$$

is Borel measurable. Observe that if $u \in \mathscr{U}$, then $u(x) \in K$.

It turns out that with any $u \in \mathscr{U}$ we can associate a weakly*-measurable mapping from $L^\infty_{w^*}(\Omega, \mathrm{rca}(K))$, such that the values of this mapping belong to $rca^1(K)$. This is one of the corollaries of the basic Theorem 3.6 in [17], see also Theorem 2 in [13]. We call them the *Young measures* and denote the set of Young measures by $\mathscr{Y}(\Omega, K)$:

$$\mathscr{Y}(\Omega,K) := \left\{ \nu = (\nu(x)) \in L^\infty_{w^*}(\Omega, \mathrm{rca}(K)) : \nu_x \in \mathrm{rca}^1(K) \text{ for a.a } x \in \Omega \right\}.$$

We will write ν_x or $(\nu_x)_{x\in\Omega}$ instead of $\nu(x)$. To emphasize the Young measure associated with particular function $u \in \mathscr{U}$, we will write ν^u. If the Young measure $(\nu_x)_{x\in\Omega}$ does not depend on the parameter $x \in \Omega$, it is called *homogeneous*.

Finally, recall that a sequence (ν_n) of bounded measures on a compact set $K \subset \mathbb{R}^l$ converges weakly* to a measure ν_0, if $\forall \beta \in C(K, \mathbb{R})$ there holds

$$\lim_{n\to\infty} \int_K \beta(k) d\nu_n(k) = \int_K \beta(k) d\nu_0(k), \tag{1}$$

see for example Theorems 15.3 and 15.11 in [18].

3. Convergent and divergent sequences of simple functions

We consider sequences of functions that are piecewise constant. If for every $x \in \Omega$ we have $u(x) = p \in \mathbb{R}^l$, where p is fixed, then $\nu^u = \delta_p$. Let a family $\{\Omega_i\}_{i=1}^n$ be a finite open partition of the set Ω, that is, the sets in this family satisfy the conditions:

(P1) $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $1 \leq i, j \leq n$;

(P2) $\displaystyle\bigcup_{i=1}^n \mathrm{cl}(\Omega_i) = \mathrm{cl}(\Omega)$;

(P3) $\forall i \in \{1, \ldots, n\}$, $\Omega_i = \mathrm{interior}\big(\mathrm{cl}(\Omega_i)\big)$,

where 'interior' stands for the topological interior of a set and 'cl' stands for the closure of a set. Denote by $diam\Omega_i$ a diameter of the partition $\{\Omega_i\}$:

$$diam\Omega_i := \max\{m_i \colon i = 1, \ldots, n\},$$

where $m_i := \mu(\Omega_i)$. We will say for brevity that a piecewise constant function u on Ω is *associated with a partition* $\{\Omega_i\}_{i=1}^n$, if it is constant on each subset Ω_i, $i = 1, 2, \ldots, n$. We can also start from considering a piecewise constant function on Ω, taking a finite number of values, each on a subset having a positive Lebesgue measure and say that this function *determines* a finite partition of Ω. Such a partition satisfies the three conditions above. It is known (see for example [19]) that the Young measure associated with u is a discrete measure and is of the form

$$\nu^u = \tfrac{1}{M} \sum_{i=1}^n m_i \delta_{p_i}, \tag{2}$$

where p_i is the value that u takes on the set Ω_i, $i = 1,2,\dots,n$. Since ν^u does not depend on the argument of the function u, it is a homogeneous Young measure.

3.1. Pointwise convergence case

Consider a piecewise constant function u_k on Ω, associated with a partition $\{\Omega_i^k\}_{i=1}^{n(k)}$:

$$u_k(x) := \sum_{i=1}^{n(k)} p_i^{(k)} \mathbf{1}_{\Omega_i^k}(x), \tag{3}$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A. Then, all the values taken by the elements of the sequence (u_k) lie in a closure of the set $K := \bigcup_{k=1}^{\infty} u_k(\Omega)$, so they are in a compact set. Assume, that this sequence is pointwise convergent to a function $u_0(x) = \sum_{i=1}^{n_0} p_i^0 \mathbf{1}_{\Omega_i^0}(x)$.

On the basis of the equation (2), and the fact that $\mathbb{R}^l$ is a separable metric space, we can say that the Young measure associated with the limit function u_0 is of the form

$$\nu^{u_0} = \tfrac{1}{M} \sum_{i=1}^{n_0} m_i^0 \delta_{p_i^0}.$$

3.2. Pointwise divergence case

Choose and fix the vectors $p_1,\dots,p_n \in \mathbb{R}^l$. Let a function u_1 on Ω be piecewise constant and takes the value p_i on the set having the respective Lebesgue measure $m_i^1 > 0, i = 1,\dots,n$. The value p_i, for a fixed i, can be taken by u_1 on different disjoint subsets of Ω, of positive Lebesgue measures summing up to m_i^1. Obviously, $\sum_{i=1}^{n} m_i^1 = M$. This function determines a finite partition $\{\Omega_{i^1}^1\}_{i^1 \in I_1}$ of the set Ω, satisfying the conditions $(P1)$-$(P3)$. The diameter of this partition is equal to $d_1 = diam \Omega_i^1$. Denote by a_1 an n-dimensional vector with the respective coordinates equal to $m_1^1,\dots,m_n^1$.

The Young measure associated with the function u_1 is of the form

$$\nu^{u_1} = \frac{1}{M} \sum_{i=1} m_i^1 \delta_{p_i}.$$

Consider now a function u_2 that takes the value p_i on the set having the respective Lebesgue measures $m_i^2 > 0, i = 1,\dots,n$. Analogously as above, the value p_i, for fixed

i, can be taken by u_2 on different disjoint subsets of Ω, and $\sum_{i=1}^{n} m_i^2 = M$. The function u_2 determines a finite partition $\{\Omega^2_{i^2}\}_{i^2 \in I_2}$ of the set Ω, satisfying the conditions $(P1)$-$(P3)$. We want the cardinality of the set I_2 of indices to be bigger than the cardinality of the set I_1. The diameter of this partition is equal to $d_2 = diam\Omega_i^2$. Denote by a_2 an n-dimensional vector with the respective coordinates equal to $m_1^2, \dots, m_n^2$.

Proceeding this way, we obtain a sequence (u_k) of piecewise constant oscillating functions taking the values $p_1, \dots, p_n$ and a sequence (a_k) of n-dimensional vectors with coordinates equal to the Lebesgue measures of the sets, on which the respective values are taken by these functions. The functions u_k determine the partitions with respective diameters d_k. Assume additionally that $\lim_{k\to\infty} d_k = 0$ and that $\lim_{k\to\infty} a_k = a_0$, where $a_0 = [m_1^0, \dots, m_n^0]$ and $m_i^0 > 0$, $i = 1, \dots, n$, with $\sum_{i=1}^{n} m_i^0 = M$.

Observe that the sequence (u_k) is not convergent to a function on Ω with values in $\mathbb{R}^l$, since the graphs of its elements 'tend' to a set $\bigcup_{i=1}^{n} (\Omega \times p_i)$ as k tends to infinity, which is not a graph of any function. However, we have the following result.

Theorem 1 *Let there be given a sequence (u_k) of functions of the form as described above. Let $(\nu_k^{u_k})$ be a sequence of Young measures associates with the respective elements of the sequence (u_k). Then $(\nu_k^{u_k})$ is weakly* convergent to a Young measure ν_0 of the form*

$$\nu_0 = \frac{1}{M} \sum_{i=1}^{n} m_i^0 \delta_{p_i}.$$

PROOF Choose and fix a function $\beta \in C(K, \mathbb{R})$. Then there holds:

$$\Big| \int_K \beta(k) d\nu^{u_k} - \int_K \beta(k) d\nu^0 \Big| = \Big| \int_K \beta(k) d\Big(\frac{1}{M} \sum_{i=1}^{n} m_i^k \delta_{p_i}\Big) - \int_K \beta(k) d\Big(\frac{1}{M} \sum_{i=1}^{n} m_i^0 \delta_{p_i}\Big) \Big| \leq$$

$$\leq \frac{1}{M} \sup_K \beta \cdot \sum_{i=1}^{n} |m_i^k - m_i^0| \xrightarrow[k\to\infty]{} 0.$$

The result then follows from the arbitrariness of the choice of β and the equation (1). ■

Example 1 Consider a partition $\{\Omega_i^{(1)}\}_{i=1}^n$ of the set Ω and assume that this partition satisfies the conditions $(P1)$-$(P3)$. Choose and fix points $p_1, \dots, p_n \in \mathbb{R}^l$ and let $K := \{p_1, \dots, p_n\}$. The set K is finite, so it is compact. The function

$$u_1(x) := \sum_{i=1}^{n} p_i \mathbf{1}_{\Omega_i^{(1)}}(x) \tag{4}$$

is piecewise constant on Ω and is associated with the partition $\{\Omega_i^{(1)}\}_{i=1}^n$.

Next, take a set $\Omega_1^{(1)}$ and consider its partition into the open sets $\Omega_{1,1}^{(2)}, \Omega_{1,2}^{(2)}, \ldots, \Omega_{1,n}^{(2)}$ in such way that for any $j = 1, 2, \ldots, n$ there holds

$$\frac{\mu\left(\Omega_{1,j}^{(2)}\right)}{\mu\left(\Omega_1^{(1)}\right)} = \frac{\mu\left(\Omega_j^{(1)}\right)}{\mu(\Omega)},$$

that is

$$\mu\left(\Omega_{1,j}^{(2)}\right) = \frac{\mu\left(\Omega_j^{(1)}\right) \cdot \mu\left(\Omega_1^{(1)}\right)}{\mu(\Omega)}. \tag{5}$$

We proceed analogously with the remaining elements of $\{\Omega_i^{(1)}\}_{i=1}^n$, obtaining an open partition

$$\Omega_{1,1}^{(2)}, \Omega_{1,2}^{(2)}, \ldots, \Omega_{1,n}^{(2)}, \Omega_{2,1}^{(2)}, \ldots, \Omega_{2,n}^{(2)}, \ldots, \Omega_{n,n}^{(2)}$$

of the set Ω, satisfying the conditions $(P1)$-$(P3)$.

Let a function u_2, associated with this partition, take the value p_1 on the sets $\Omega_{1,1}^{(2)}, \Omega_{2,1}^{(2)}, \ldots, \Omega_{n,1}^{(2)}$, the value p_2 on the sets $\Omega_{1,2}^{(2)}, \Omega_{2,2}^{(2)}, \ldots, \Omega_{n,2}^{(2)}$, and so on up to p_n. Then u_2 takes the value p_1 on the set having a Lebesgue measure equal to $\mu\left(\Omega_1^{(1)}\right)$, the value p_2 on the set having Lebesgue measure equal to $\mu\left(\Omega_2^{(1)}\right)$, and so on.

Continuing this way, we obtain a sequence (u_k) of piecewise constant functions, such that each element of this sequence takes the value p_1 on the set having Lebesgue measure equal to $\mu\left(\Omega_1^{(1)}\right)$,..., the value p_n on the set having a Lebesgue measure equal to $\mu\left(\Omega_n^{(1)}\right)$. The diameters of the respective partitions form a convergent to zero sequence of reals. Furthermore, the sequence (u_k) has the properties:

- it is divergent in L^∞;
- the sequence (a_k) of the n-dimensional vectors with the coordinates equal to the Lebesgue measure of the sets, on which u_k takes the respective values, is constant;
- the sequence of Young measures associated with the respective elements of (u_k) is constant. Each element of this sequence of measures is of the form

$$\nu^{u_k} = \frac{1}{M} \sum_{i=1}^{n} \mu\left(\Omega_i^{(1)}\right) \delta_{p_i}.$$

Thus the sequence $\left(\nu^{u_k}\right)$ is trivially convergent in the norm topology of $rca(K)$, so it is all the more weakly* convergent. This Young measure is also a generalized weak* limit of the sequence (u_k). □

4. Conclusion

Homogeneous discrete Young measures appear for example in optimization, when searching the minimizers of the multiwell problems, see for instance [2], [17] or [20] and the references therein. Usually, calculating an explicit form of generalized limit of a function sequence, which in our case is a measure-valued mapping, is a difficult task and involves advanced methods of functional analysis. In this article, we propose a method that generalizes and simplifies, at least to some extent, calculating those limits in some specific, but not rare, cases. This may be useful in some applications, in particular in engineering.

It seems useful to consider the sequences of Young measures associated with the respective elements of the function sequence of interest, and to investigate the existence and the form of the weak* limits of this sequence of measures.

It also turns out that we can expect the strong convergence of the sequence of measures only in the case, when the respective sequence of Young measures is constant. When it is not, the norms $\|\nu^{u_k} - \nu^{u_0}\|_{rca(K)}$ does not tend to zero as $k \to \infty$. It follows from the fact that the total variation norm of a discrete measure with the component Dirac measures supported at different points is the sum of the absolute values of the respective coefficients.

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