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ON THE DARBOUX VECTOR OF A NEW FRACTIONAL FRAME

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Abstract. Fractional derivatives are useful tools for many applications in different branch of science such as optics and engineering. In this paper, the $\wedge$-fractional frame that is obtained along a space curve by using the $\wedge$-fractional derivative is being examined in Euclidean $\mathbb{E}^3$ space. In addition, the Darboux vector of the $\wedge$-fractional Frenet frame is constructed. Then the curvatures of the standard Frenet frame, the $\wedge$-fractional Frenet frame and the $\wedge$-fractional Darboux vector are compared geometrically.

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1. Introduction

Fractional calculus has been an area of interest since the 17th century and has been studied by great mathematicians such as Riemann, Bernoulli, Leibniz, Euler and others [1–3]. Integer order derivatives and integrals have clear physical and geometric interpretations to solve simplified applied problems. However, fractional order derivatives and integrals have a rapidly growing field both in theory and in applications to real-world problems, but not yet at the desired level. Presently, this field is an effective method to better explain the real-world model. The fractional derivative is assumed to have wider geometric and physical properties than the known integer order derivative. Many physical papers have been published in response to the expectations [4–9]. Scientists have wondered about the geometric interpretation of the fractional order derivative and integral. Therefore, various approaches have been presented. These approaches can be found in [10–17].

Although the differences of the fractional derivative are an important factor in these approaches, most of the definitions of the fractional derivative do not fulfill some of the requirements of differential topology, such as the Leibniz rule [14].

[1] This article is extracted from my master thesis entitled "Uzay Eğrisi Boyunca $\wedge$-Frenet çatısı", supervised by Mustafa DEDE (Master's Thesis Kilis 7 Aralık University, Kilis, Turkey, 2023)

However, the $\wedge$-fractional derivative operator corresponding to the $\wedge$-fractional space satisfies the requirements of differential topology. Therefore, it is a tool for many applications. Moreover, $\wedge$-fractional ordinary and partial differential equations are discussed in [18].

Along a space curve $r(s)$, the variation of the Frenet frame $\{T(s), N(s), B(s)\}$ is indicated by the Darboux vector D, which is a measure of the instantaneous rate of change of each of the vectors $\{T, N, B\}$ and given by

$$\frac{dT}{ds} = D \times T, \frac{dN}{ds} = D \times N, \frac{dB}{ds} = D \times B, \tag{1}$$

where s is the arc length along $r(s)$ [19–22].

The Darboux $D = \tau T + \kappa B$ vector offers a practical way to interpret the curvature κ and torsion τ in differential geometry. Curvature represents the degree of rotation of the Frenet frame around the binormal unit vector, whereas torsion indicates the amount of rotation of the Frenet frame around the tangent unit vector [19]. So curvature and torsion are essential tools for understanding the behaviour of the frame.

The rate of instantaneous rotation is given by

$$\|D\| = \sqrt{\tau^2 + \kappa^2} \tag{2}$$

The Frenet frame is a standard tool for understanding the concepts underlying differential geometry [19]. However, for geometric modeling applications, it is controversial that the Frenet frame is the best frame to choose since it has a strong rotation around the tangent vector. To avoid this negative situation, the so-called rotation minimizing frame (RMF) has been introduced and studied in the literature [19–22].

This article consists of 4 parts. In the first part, the importance and rapid development of the fractional derivative in various fields are mentioned. In the second part, the basic concepts of fractional derivatives and integrals are explained and devoted to $\wedge$-fractional Frenet frame corresponding to the $\wedge$-fractional derivative. The third part contains the $\wedge$-Darboux vector corresponding to the $\wedge$-fractional Frenet frame and curvatures. The last section is devoted to the conclusion.

2. Preliminaries

In this section, some fundamental concepts and results of fractional analysis are introduced. In addition, the $\wedge$-fractional derivative that is obtained with the help of the Riemann-Liouville fractional derivative, and the integral is expressed.

For a function $f(\upsilon) \in L^1[a,b]$ that is continuous and differentiable, the left and right fractional integrals of order γ are defined as follows:

$$(I^{\gamma}_{a^+} f)(\upsilon) = \frac{1}{\Gamma(\gamma)} \int_a^{\upsilon} (\upsilon - t)^{\gamma - 1} f(t) dt, \quad \upsilon > a \tag{3}$$

$$(I_{b^-}^{\gamma} f)(\upsilon) = \frac{1}{\Gamma(\gamma)} \int_{\upsilon}^{b} (t-\upsilon)^{\gamma-1} f(t)dt, \quad \upsilon < b \tag{4}$$

where $\Gamma(\gamma) = \int_0^{\infty} e^{-\upsilon}\upsilon^{\gamma-1}d\upsilon$ is Gamma function [3]. Moreover, for polynomial functions $f(\upsilon) = \upsilon^k$ is expressed in the form

$$I_{\upsilon}^{\gamma}\upsilon^k = \frac{\Gamma(k+1)}{\Gamma(\gamma+k+1)}\upsilon^{\gamma+k} \tag{5}$$

and for functions $f(\upsilon)$ given in the interval [a,b], each of the expressions

$$(D_{a^+}^{\gamma} f)(\upsilon) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{d\upsilon} \int_a^{\upsilon} (\upsilon-t)^{-\gamma} f(t)dt \tag{6}$$

$$(D_{b^-}^{\gamma} f)(\upsilon) = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{d\upsilon} \int_{\upsilon}^{b} (t-\upsilon)^{-\gamma} f(t)dt \tag{7}$$

is called a fractional derivative of order γ, $0 < \gamma < 1$. In addition, there is a connection between Riemann-Liouville fractional derivatives and integrals, as follows [3].

$$D_a^{\gamma}(I_a^{\gamma} f(\upsilon)) = f(\upsilon) \tag{8}$$

Furthermore, the Riemann-Liouville fractional derivative is defined by

$$D_a^{\gamma} f(\upsilon) = \frac{d}{d\upsilon}(I_{\upsilon}^{1-\gamma} f(\upsilon)) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{d\upsilon} \int_a^{\upsilon} (\upsilon-t)^{-\alpha} f(t)dt \tag{9}$$

Although fractional derivatives and integrals are useful tools for describing real-world problems, it is quite difficult to construct their differential geometry. Because many definitions of fractional derivatives do not satisfy certain properties, such as linearity and the Leibniz product rule. The $\wedge$-fractional derivative which is introduced below, is a convenient tool for constructing differential geometry (see [14]).

Let $f(\upsilon)$ be a differentiable function for all $\upsilon \in \mathbb{R}$. The $\wedge$-fractional derivative ($\wedge$-FD) is defined as:

$${}_{a}^{\wedge}D_{\upsilon}^{\gamma} f(\upsilon) = \frac{{}_{a}^{RL}D_{\upsilon}^{\gamma} f(\upsilon)}{{}_{a}^{RL}D_{\upsilon}^{\gamma}(\upsilon)} \tag{10}$$

Considering the definition of Riemann-Liouville fractional derivative given by Equation (9) and Equation (10) can be expressed as follows [14]

$${}_{a}^{\wedge}D_{\upsilon}^{\gamma} f(\upsilon) = \frac{\dfrac{d_a I_{\upsilon}^{1-\gamma} f(\upsilon)}{d\upsilon}}{\dfrac{d_a I_{\upsilon}^{1-\gamma}(\upsilon)}{d\upsilon}} = \frac{d_a I_{\upsilon}^{1-\gamma} f(\upsilon)}{d_a I_{\upsilon}^{1-\gamma}(\upsilon)} \tag{11}$$

where if $\xi = {}_aI_\upsilon^{1-\gamma}(\upsilon)$ and $F(\xi) = {}_aI_\upsilon^{1-\gamma} f(\upsilon)$ is taken, $\wedge$-FD is defined as traditional a derivative in $(\xi, F(\xi))$ fractional space. The fractional differential geometry is defined as conventional differential geometry in the $\wedge$-fractional space $(\xi, F(\xi))$, with the following derivative definition,

$$ {}_a^{\wedge}D_\upsilon^{\gamma} f(\upsilon) = \frac{dF(\xi)}{d\xi} \tag{12} $$

In addition,

$$ {}_a^{RL}D_\upsilon^{1-\gamma}({}_aI_\upsilon^{1-\gamma} f(\upsilon)) = f(\upsilon) $$

is very important as it pulls back various functions from the $\wedge$-fractional space to the standard space. This definition is obtained in three-dimensional Euclidean spaces as follows [14].

Let $\alpha(\upsilon) = (\upsilon, f(\upsilon), g(\upsilon))$ be a space curve in the Euclid space. Considering equations $\xi = {}_0I_\upsilon^{1-\gamma}\upsilon$, $F(\xi) = {}_0I_\upsilon^{1-\gamma} f(\upsilon)$ and $G(\xi) = {}_0I_\upsilon^{1-\gamma} g(\upsilon)$, then is constructed a new curve $\tilde{\alpha}(\xi) = (\xi, F(\xi), G(\xi))$ in the conjugate $\wedge$-space. For the curve $\wedge$-fractional $\tilde{\alpha}(\xi) : I \to \mathbb{R}^3$, the $\wedge$-fractional tangent vector, $\wedge$-fractional binormal vector and $\wedge$-fractional normal vector are defined as follows,

$$ T_\wedge(\xi) = \frac{\tilde{\alpha}'(\xi)}{\|\tilde{\alpha}'(\xi)\|}, \quad B_\wedge(\xi) = \frac{\tilde{\alpha}'(\xi) \wedge \tilde{\alpha}''(\xi)}{\|\tilde{\alpha}'(\xi) \wedge \tilde{\alpha}''(\xi)\|}, \quad N_\wedge(\xi) = B_\wedge(\xi) \wedge T_\wedge(\xi) \tag{13} $$

Furthermore, the $\wedge$-fractional curvature and torsion functions are defined as

$$ \kappa_\wedge(\xi) = \frac{\|\tilde{\alpha}'(\xi) \wedge \tilde{\alpha}''(\xi)\|}{\|\tilde{\alpha}'(\xi)\|^3} \tag{14} $$

$$ \tau_\wedge(\xi) = \frac{< \tilde{\alpha}'(\xi), \tilde{\alpha}''(\xi) \wedge \tilde{\alpha}'''(\xi) >}{\|\tilde{\alpha}'(\xi) \wedge \tilde{\alpha}''(\xi)\|^2} \tag{15} $$

where " $'$ " denotes the derivatives respect to ξ [14]. In addition to these, Serret-Frenet equations in $\wedge$-fractional space are expressed as,

$$ \begin{bmatrix} T_\wedge'(\xi) \\ N_\wedge'(\xi) \\ B_\wedge'(\xi) \end{bmatrix} = v_\wedge(\xi) \begin{bmatrix} 0 & \kappa_\wedge(\xi) & 0 \\ -\kappa_\wedge(\xi) & 0 & \tau_\wedge(\xi) \\ 0 & -\tau_\wedge(\xi) & 0 \end{bmatrix} \begin{bmatrix} T_\wedge(\xi) \\ N_\wedge(\xi) \\ B_\wedge(\xi) \end{bmatrix} \tag{16} $$

where $v_\wedge(\xi) = \|\tilde{\alpha}'(\xi)\|$ is a fractional velocity function [14].

3. The $\wedge$-fractional Darboux vector

In this section, the geometric interpretation of curvature $\kappa_\wedge$ and torsion $\tau_\wedge$ by using the $\wedge$-fractional Darboux vector is explained.

The $\wedge$-fractional Darboux vector corresponding to the fractional space $(\xi, F(\xi), G(\xi))$ defined by the $\wedge$-Frenet frame $\{T_\wedge(\xi), N_\wedge(\xi), B_\wedge(\xi)\}$ provides the following properties,

$$\begin{aligned} \mathbb{W}_\wedge(\xi) \times T_\wedge(\xi) &= T'_\wedge(\xi) \\ \mathbb{W}_\wedge(\xi) \times N_\wedge(\xi) &= N'_\wedge(\xi) \\ \mathbb{W}_\wedge(\xi) \times B_\wedge(\xi) &= B'_\wedge(\xi) \end{aligned} \tag{17}$$

Moreover, the $\wedge$-Darboux vector $\mathbb{W}_\wedge(\xi)$ can be written by using the $\wedge$-Frenet bases vector as follows,

$$\mathbb{W}_\wedge(\xi) = w_1 T_\wedge(\xi) + w_2 N_\wedge(\xi) + w_3 B_\wedge(\xi) \tag{18}$$

Vectorial multiplication of both sides of the equation (18) by $T_\wedge(\xi)$ gives,

$$\begin{gathered} \mathbb{W}_\wedge(\xi) \times T_\wedge(\xi) = -w_2 B_\wedge(\xi) + w_3 N_\wedge(\xi) \\ \Rightarrow w_2 = 0, w_3 = v_\wedge(\xi)\kappa_\wedge(\xi) \end{gathered} \tag{19}$$

Similarly, if both sides of the equation (18) are multiplied vectorially by $N_\wedge(\xi)$ and $B_\wedge(\xi)$, one obtains

$$w_1 = v_\wedge(\xi)\tau_\wedge(\xi), w_2 = 0, w_3 = v_\wedge(\xi)\kappa_\wedge(\xi) \tag{20}$$

Based on these results, the $\wedge$-Darboux vector is expressed as follows,

$$\mathbb{W}_\wedge(\xi) = v_\wedge(\xi)\tau_\wedge(\xi)T_\wedge(\xi) + v_\wedge(\xi)\kappa_\wedge(\xi)B_\wedge(\xi) \tag{21}$$

The magnitude of the $\wedge$-Darboux vector is obtained by

$$\|\mathbb{W}_\wedge(\xi)\| = v_\wedge(\xi)\sqrt{\tau^2_\wedge(\xi) + \kappa^2_\wedge(\xi)} \tag{22}$$

Example 1 Let us compute the $\wedge$-Frenet frame $\{T_\wedge, N_\wedge, B_\wedge\}$ and the $\wedge$-fractional curvature and torsion functions of the curve given by

$$\begin{gathered} \alpha : \mathbb{R} \to \mathbb{E}^3 \\ \upsilon \to \alpha(\upsilon) = (\upsilon, \upsilon^2, \upsilon^3) \end{gathered} \tag{23}$$

Inspired by Eqs. (13), the $\wedge$-fractional curve is expressed in the following form,

$$\tilde{\alpha}(\upsilon) = \left({}_0I_\upsilon^{1-\gamma}\upsilon,\ {}_0I_\upsilon^{1-\gamma}\upsilon^2,\ {}_0I_\upsilon^{1-\gamma}\upsilon^3\right) \tag{24}$$

Since the fractional integral of the polynomial function υ^k is calculated as follows,

$${}_0I_\upsilon^\eta \upsilon^k = \frac{\Gamma(k+1)}{\Gamma(\eta+k+1)}\upsilon^{\eta+k} \tag{25}$$

the components of the curve $\tilde{\alpha}(\upsilon)$ are found

$$\begin{aligned} {}_0I_\upsilon^{1-\gamma}\upsilon &= \frac{\Gamma(2)}{\Gamma(3-\gamma)}\upsilon^{2-\gamma} \\ {}_0I_\upsilon^{1-\gamma}\upsilon^2 &= \frac{\Gamma(3)}{\Gamma(4-\gamma)}\upsilon^{3-\gamma} \\ {}_0I_\upsilon^{1-\gamma}\upsilon^3 &= \frac{\Gamma(4)}{\Gamma(5-\gamma)}\upsilon^{4-\gamma} \end{aligned} \tag{26}$$

The $\wedge$-fractional $\tilde{\alpha}(\xi)$ is expressed as following,

$$\tilde{\alpha}(\xi) = (\xi, F(\xi), G(\xi)) \tag{27}$$

where

$$\xi = \frac{\Gamma(2)}{\Gamma(3-\gamma)}\upsilon^{2-\gamma} \tag{28}$$

By using equation (27), we have

$$\upsilon = \Gamma(3-\gamma)^{\frac{1}{2-\gamma}}\xi^{\frac{1}{2-\gamma}} \tag{29}$$

Combining equations (26) and (29), it becomes

$$\begin{aligned} F(\xi) &= \frac{2}{(3-\gamma)}\Gamma(3-\gamma)^{\frac{1}{2-\gamma}}\xi^{\frac{3-\gamma}{2-\gamma}} \\ G(\xi) &= \frac{6}{(4-\gamma)(3-\gamma)}\Gamma(3-\gamma)^{\frac{2}{2-\gamma}}\xi^{\frac{4-\gamma}{2-\gamma}} \end{aligned} \tag{30}$$

Finally, $\wedge$-fractional curve $\tilde{\alpha}(\xi)$ is found as

$$\tilde{\alpha}(\xi) = \left(\xi, \frac{2\Gamma(3-\gamma)^{\frac{1}{2-\gamma}}\xi^{\frac{3-\gamma}{2-\gamma}}}{(3-\gamma)}, \frac{6\Gamma(3-\gamma)^{\frac{2}{2-\gamma}}\xi^{\frac{4-\gamma}{2-\gamma}}}{(4-\gamma)(3-\gamma)}\right)$$

Now lets calculate the $\wedge$-Frenet frame of the curve $\tilde{\alpha}(\xi)$. The first order differential according to the ξ parameter of fractional curve $\tilde{\alpha}(\xi)$ is obtained as

$$\tilde{\alpha}'(\xi) = \left(1, \frac{2\Gamma(3-\gamma)^{\frac{1}{2-\gamma}}\xi^{\frac{1}{2-\gamma}}}{2-\gamma}, \frac{6\Gamma(3-\gamma)^{\frac{2}{2-\gamma}}\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)(2-\gamma)}\right) \tag{31}$$

For the sake of simplicity, let us take $\psi = \dfrac{\Gamma(3-\gamma)^{\frac{1}{2-\gamma}}}{2-\gamma}$, so we have

$$\tilde{\alpha}'(\xi) = \left(1, 2\psi\xi^{\frac{1}{2-\gamma}}, \frac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{3-\gamma}\right) \tag{32}$$

Similarly, the second and third order derivatives of the fractional curve are calculated as,

$$\begin{aligned} \tilde{\alpha}''(\xi) &= \left(0, \frac{2\psi\xi^{\frac{\gamma-1}{2-\gamma}}}{(2-\gamma)}, \frac{12\psi^2\xi^{\frac{\gamma}{2-\gamma}}}{(3-\gamma)}\right) \\ \tilde{\alpha}'''(\xi) &= \left(0, \frac{2(\gamma-1)\psi\xi^{\frac{2\gamma-3}{2-\gamma}}}{(2-\gamma)^2}, \frac{12\gamma\psi\xi^{\frac{2\gamma-2}{2-\gamma}}}{(3-\gamma)(2-\gamma)}\right) \end{aligned} \tag{33}$$

The fractional tangent vector, binormal vector and normal vector of the curve $\tilde{\alpha}(\xi)$ is found as follows:

$$T_\wedge(\xi) = \frac{\left(1, 2\psi\xi^{\frac{1}{2-\gamma}}, \dfrac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)}\right)}{\sqrt{1 + \left(2\psi\xi^{\frac{1}{2-\gamma}}\right)^2 + \left(\dfrac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)}\right)^2}} \tag{34}$$

$$B_\wedge(\xi) = \frac{\left(\dfrac{12\psi^3\xi^{\frac{\gamma+1}{2-\gamma}}}{(3-\gamma)}, \dfrac{-12\psi^2\xi^{\frac{\gamma}{2-\gamma}}}{(3-\gamma)}, \dfrac{2\psi\xi^{\frac{\gamma-1}{2-\gamma}}}{(2-\gamma)}\right)}{\sqrt{\left(\dfrac{12\psi^3\xi^{\frac{\gamma+1}{2-\gamma}}}{(3-\gamma)}\right)^2 + \left(\dfrac{-12\psi^2\xi^{\frac{\gamma}{2-\gamma}}}{(3-\gamma)}\right)^2 + \left(\dfrac{2\psi\xi^{\frac{\gamma-1}{2-\gamma}}}{(2-\gamma)}\right)^2}} \tag{35}$$

and

$$N_\wedge(\xi) = \frac{(N_{\wedge 1}(\xi), N_{\wedge 2}(\xi), N_{\wedge 3}(\xi))}{\sqrt{M(\xi)N(\xi)}} \tag{36}$$

where

$$\begin{aligned} N_{\wedge 1}(\xi) &= \frac{36(2-\gamma)^2\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)} + 2\gamma - 6 \\ N_{\wedge 2}(\xi) &= 2(3-\gamma)\psi^{-1}\xi^{\frac{-1}{2-\gamma}} - \frac{36(2-\gamma)^2\psi^3\xi^{\frac{3}{2-\gamma}}}{(3-\gamma)} \end{aligned} \tag{37}$$

$$N_{\wedge 3}(\xi) = \frac{36(2-\gamma)^2\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)} + 2\gamma - 6$$

$$M(\xi) = 36(2-\gamma)^2\psi^2\xi^{\frac{2}{2-\gamma}} + 4(2-\gamma)^2 + (3-\gamma)^2\psi^{-2}\xi^{\frac{-2}{2-\gamma}} \tag{38}$$

$$N(\xi) = 1 + (2\psi\xi^{\frac{1}{2-\gamma}})^2 + (\frac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)})^2$$

The fractional curvature and torsion of the curve $\tilde{\alpha}(\xi)$, respectively, are calculated as:

$$\kappa_{\wedge}(\xi) = \frac{\sqrt{\left(\frac{12\psi^3\xi^{\frac{\gamma+1}{2-\gamma}}}{(3-\gamma)}\right)^2 + \left(\frac{-12\psi^2\xi^{\frac{\gamma}{2-\gamma}}}{(3-\gamma)}\right)^2 + \left(\frac{2\psi\xi^{\frac{\gamma-1}{2-\gamma}}}{(2-\gamma)}\right)^2}}{\left(\sqrt{1 + \left(2\psi\xi^{\frac{1}{2-\gamma}}\right)^2 + \left(\frac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)}\right)^2}\right)^3} \tag{39}$$

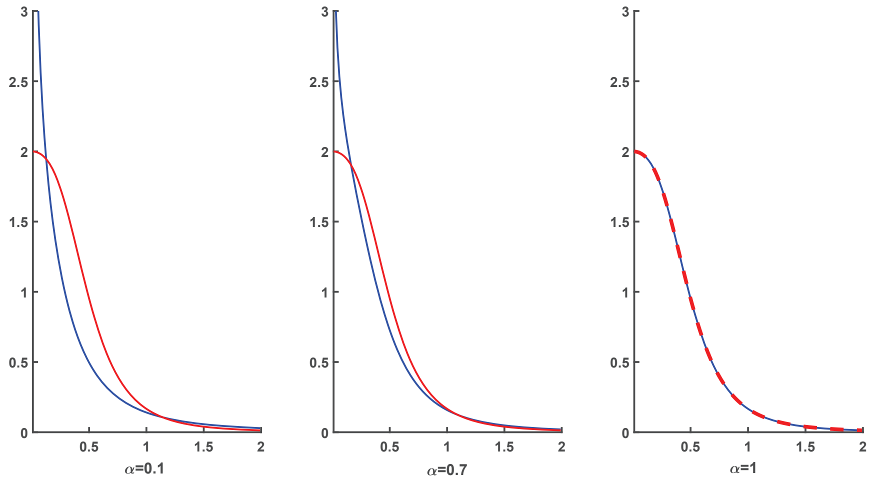


Fig. 1. The fractional curvature $\kappa_{\wedge}(blue)$ and curvature $\kappa(red)$

In Figure 1, it is easy to see that when the γ value approaches 1, the fractional curvature approaches the curvature. However, as the γ value approaches 0, the curvature becomes smaller in a certain range.

$$\tau_{\wedge}(\xi) = \frac{\frac{24\psi\xi^{\frac{3\gamma-3}{2-\gamma}}\left(1-\gamma+\gamma(2-\gamma)^2\psi^2\right)}{(3-\gamma)(2-\gamma)^4}}{\left(\frac{12\psi^3\xi^{\frac{\gamma+1}{2-\gamma}}}{(3-\gamma)}\right)^2 + \left(\frac{-12\psi^2\xi^{\frac{\gamma}{2-\gamma}}}{(3-\gamma)}\right)^2 + \left(\frac{2\psi\xi^{\frac{\gamma-1}{2-\gamma}}}{(2-\gamma)}\right)^2} \tag{40}$$

Similarly, the same geometric interpretation can be made for torsion as shown in Figure 2.

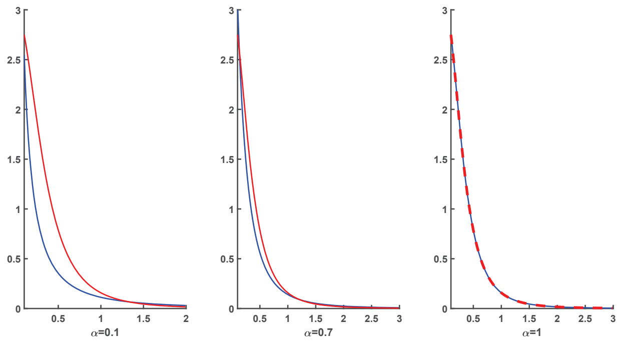


Fig. 2. The fractional torsion $\tau_{\wedge}(blue)$ and torsion $\tau(red)$

In addition to the $\wedge$-fractional Darboux vector of the fractional curve, $\tilde{\alpha}(\xi)$ is found as,

$$
\begin{aligned}
&\mathbb{W}_{\wedge}(\xi) = \\
&\frac{\dfrac{6(3-\gamma)\psi^{-1}\xi^{\frac{\gamma-1}{2-\gamma}}\left(1-\gamma+\gamma(2-\gamma)^2\psi^2\right)}{(2-\gamma)^2}\left(1, 2\psi\xi^{\frac{1}{2-\gamma}}, \dfrac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)}\right)}{\left[\left(6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}\right)^2+\left(-6\psi\xi^{\frac{1}{2-\gamma}}\right)^2+(3-\gamma)^2\right]}+ \\
&\frac{\left(\dfrac{12\psi^3\xi^{\frac{\gamma+1}{2-\gamma}}}{(3-\gamma)}, \dfrac{-12\psi^2\xi^{\frac{\gamma}{2-\gamma}}}{(3-\gamma)}, \dfrac{2\psi\xi^{\frac{\gamma-1}{2-\gamma}}}{(2-\gamma)}\right)}{\left[1+\left(2\psi\xi^{\frac{1}{2-\gamma}}\right)^2+\left(\dfrac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)}\right)^2\right]^2}
\end{aligned}
\tag{41}
$$

The magnitude of the $\wedge$-fractional Darboux vector is that,

$$
\begin{aligned}
&\|\mathbb{W}_{\wedge}(\xi)\| = \\
&\frac{\sqrt{1+\left(2\psi\xi^{\frac{1}{2-\gamma}}\right)^2+\left(\frac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)}\right)^2}\left(\frac{6(3-\gamma)\psi^{-1}\xi^{\frac{\gamma-1}{2-\gamma}}\left(1-\gamma+\gamma(2-\gamma)^2\psi^2\right)}{(2-\gamma)^2}\right)^2}{\left(\left(6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}\right)^2+\left(-6\psi\xi^{\frac{1}{2-\gamma}}\right)^2+(3-\gamma)^2\right)^2}+ \\
&\frac{\left(\frac{12\psi^3\xi^{\frac{\gamma+1}{2-\gamma}}}{(3-\gamma)}\right)^2+\left(\frac{-12\psi^2\xi^{\frac{\gamma}{2-\gamma}}}{(3-\gamma)}\right)^2+\left(\frac{2\psi\xi^{\frac{\gamma-1}{2-\gamma}}}{(2-\gamma)}\right)^2}{\left(\sqrt{1+\left(2\psi\xi^{\frac{1}{2-\gamma}}\right)^2+\left(\frac{6(2-\gamma)\psi^2\xi^{\frac{2}{2-\gamma}}}{(3-\gamma)}\right)^2}\right)^5}
\end{aligned}
\tag{42}
$$

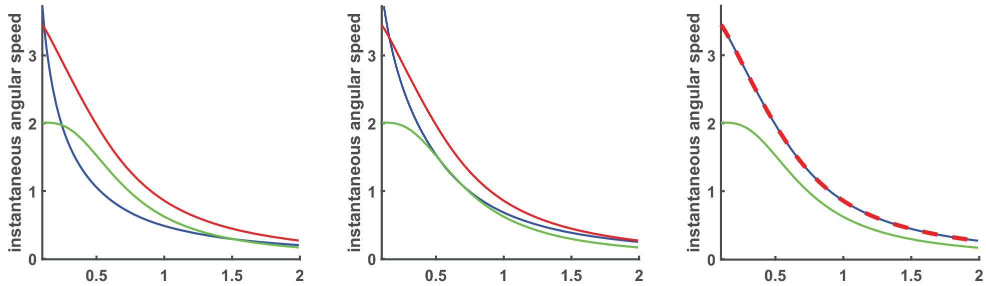
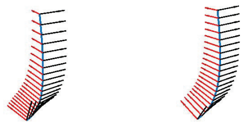
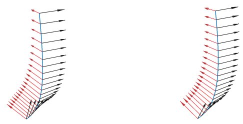


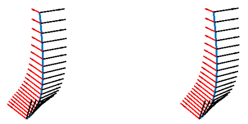
Fig. 3. Comparison of the instantaneous angular speed of the $\wedge$-Frenet frame (blue), Frenet frame (red), Bishop frame (green)



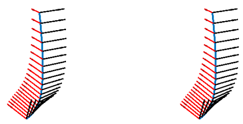
(a) $\gamma = \frac{1}{100}$



(b) $\gamma = \frac{1}{2}$



(c) $\gamma = \frac{9}{10}$



(d) $\gamma = 1$

Fig. 4. The Frenet frame (left) and $\wedge$ Frenet frame along the curve the normal-plane vectors are shown

Figure 3 illustrates three special cases of the instantaneous angular speed of the Λ-Frenet frame (blue), Frenet frame (red), Bishop frame (green) where $\gamma = 0.1$, $\gamma = 0.7$ and $\gamma = 1$ on the interval $[1,2]$ are considered.

Finally, the behaviour of the normal-plane vectors (the normal vectors (red) and binormal vectors (black)) of the Frenet frame with the Λ-fractional Frenet frame is compared in Figure 4.

4. Conclusions

In this study, the norm of the $\wedge$-fractional Darboux vector obtained with the help of the $\wedge$-fractional derivative gives better results than the norm of the Darboux vector obtained from the standard Frenet and Bishop frame in a certain range. However, the $\wedge$-fractional Frenet frame is more difficult to construct than the standard Frenet frame. As expected, as γ approaches one, the $\wedge$-Frenet frame approaches to the standard Frenet frame.

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