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SOLUTION OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS USING LEAST SQUARES AND SHIFTED LEGENDRE METHODS

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Abstract. In recent years, fractional calculus (FC) has filled in a hole in traditional calculus in terms of the effect of memory, which lets us know things about the past and present and guess what will happen in the future. It is very important to have this function, especially when studying biological models and integral equations. This paper introduces developed mathematical strategies for understanding a direct arrangement of fractional integro-differential equations (FIDEs). We have presented the least squares procedure and the Legendre strategy for discussing FIDEs. We have given the form of the Caputo concept fractional order operator and the properties. We have presented the properties of the shifted Legendre polynomials. We have shown the steps of the technique to display the solution. Some test examples are given to exhibit the precision and relevance of the introduced strategies. Mathematical outcomes show that this methodology is a comparison between the exact solution and the methods suggested. To show the theoretical results gained, the simulation of suggested strategies is given in eye-catching figures and tables. Program Mathematica 12 was used to get all of the results from the techniques that were shown.

MSC 2010: *41A10, 65N12, 65N35*
Keywords: *fractional integro-differential equations, Legendre polynomial, least squares technique, residual equation, weight function, Caputo derivative*

1. Introduction

Numerous studies have been carried out with FC to simulate real-world issues and to obtain a deeper comprehension of the influence of genetic characteristics and memory on certain epidemiological models [1, 2]. Integro-differential and differential conditions of fractional requests emerge in numerous physical and design issues, for example, liquid mechanics, viscoelasticity, dispersion cycles, science, etc. [1, 3]. Ahmed and Salh in [4], introduced a modified Taylor matrix approach for resolving linear integer fractional problems of the Volterra sort. Bhrawy and Alofi in [5], presented the operational matrix of fractional integration for shifted Chebyshev polynomials. Alquran in [6], introduced the incredible fractional Maclaurin series to resolve many fractional mathematical issues that come up in physics and engineering. Khader et al. in [7], were proposed an effective numerical approach to calculate the fractional diffusion equation. Pakchin and Mazraeh use a variant of He's variational iteration approach in [8] to examine exact solutions for a few FIDEs with nonlocal boundary conditions. Jaradat et al. in [9], presented an analytical simulation of the synergy of proportionate time-delayed spatial-temporal memory indices. Zurigat et al. examine ahomotopy analytic technique for fractional integro-differential equation systems in [10]. His homotopy perturbation approach, which Nadjafi and Gorbani apply in [11], is a useful tool for resolving nonlinear integral and integro--differential equations. His homotopy perturbation approach for integro-differential equation systems is described by Biazar et al. in [12]. The fractional differential transform approach to solving FIDEs by Arikoglu and Ozkol is shown in [13]. Rawashdeh used the collocation method in [14] to solve FIDEs numerically. Huang et al. use the Taylor expansion approach in [15] to approximate the solution of FIDEs. The CAS (Cosine and Sine) wavelets approach for solving fractional order nonlinear Fredholm integro-differential equations was presented by Saeedi et al. in [16]. Maleknejad et al. provided a numerical solution by block pulse functions for a system of second-kind integral equations [17]. Alquran et al. in [18], introduced the use of residual power series and Laplace transforms in combination to solve n-dimensional fractional nonlinear problems. According to Ali et al., [2], they found the exact and close solutions for the fractional diffusive Predator-Prey model using the conformable Caputo. Bell provides unique functions for engineers and scientists [19]. Fractional integrals and derivative notions and applications are published by Samko et al. in [3]. Mahdy solves FIDEs in [20] by use of numerical investigations. Amer et al. use the Hermite spectral collocation method and the Sumudu transform method to solve FIDEs [21]. Mahdy and Mohamed employed numerical research [22] to solve a set of linear FIDEs by shifting Chebyshev polynomials and the least squares approach. Mahdy and Shwayye in [23] discuss the numerical solution of FIDEs employing the shifted Laguerre polynomials pseudo-spectral approach and the least squares method. In [24], Mohammed applies the shifted Chebyshev polynomial and the least squares style to numerically solve FIDEs. Mahdy et al. presented a computer strategy in [26] for resolving mixed Volterra Fredholm integral equations in three dimensions. Mahdy et al. in [27] provided an algorithmic method for mixed integral equations with unique kernels at the second kind. Lucas polynomials were used by Mahdy and Mohamed in [28] to estimate their responses to Cauchy integral equations. Chelyshkov's polynomials method for solution 2-dim nonlinear Volterra integral equations of the first sort is described by Mahdy et al. in [29]. A portion of these mathematical strategies is Adomian's disintegration strategy, variety cycle technique, homotopy investigation strategy, differential change strategy, operational lattices, and nonstandard limited contrast in [10]. Fractional differential equations (FDEs) are usually used to illustrate models in the realm of thermoelasticity, organic standards, and frameworks with a memory present in a few physical occurrences. FDEs have been applied to demonstrate how flexible frames and disease contamination can diminish reasonably in two phases, albeit more slowly. Number requests are not as useful as FDEs when displaying intricate models that include real Marvel.

This work focuses on the numerical solution of the FIDEs that follows:

$$D^{\gamma} z_j(y) = g_j(y) + \int_0^1 k_j(y,t) \left(\sum_{i=1}^{r} \gamma_{ik} z_k(t) \right) dt, \tag{1}$$

at initial conditions

$$z_j^{(i)}(y_0) = z_{ij} \quad i = 1, ..., n, \quad n-1 < \gamma \leq r, r \in \mathbb{N},$$

where $D^{\gamma} z_j(y)$ signalizes the γ^{th} Caputo fractional order of $z_j(y), g_j(y), k_j(y,t)$ have fixed functions, the real y,t in $[0,1]$ and $z_j(y)$ has the obscure functions to be specific.

2. Definitions and fundamentals

This section provides some fundamental concepts and properties of FC theory that are required to formulate the problem.

2.1. Caputo fractional order

The form of the Caputo fractional order operator D^{γ} of order γ is noted:

$$D^{\gamma} g(y) = \frac{1}{\Gamma(m-\gamma)} \int_0^y \frac{g^{(m)}(t)}{(y-t)^{\gamma-m+1}} \, dt, \quad \gamma > 0,$$

where: $m-1 < \gamma$; $m \in \mathbb{N}$, $y > 0$.

The operator of the fractional Caputo derivative is an operation linear, identical to differentiation order-integer:

$$D^{\gamma}(\lambda_1 f(z) + \lambda_2 g(z)) = \lambda_1 D^{\gamma} f(z) + \lambda_2 D^{\gamma} g(z),$$

where, λ_1 and λ_2 are parameters. In accordance with Caputo's order, we have

$$D^{\gamma} O = 0, \qquad O \text{ is a paramter,} \tag{2}$$

$$D^{\gamma} z^r = \begin{cases} 0, & \text{for } r \in \mathbb{N}_0 \text{ and } r < \ulcorner \gamma \urcorner; \\ \dfrac{\Gamma(r+1)}{\Gamma(r+1-\gamma)} z^{r-\gamma} & \text{for } r \in \mathbb{N}_0 \text{ and } r \geq \ulcorner v \urcorner. \end{cases} \tag{3}$$

The ceiling role $\ulcorner\gamma\urcorner$ is used to symbolize the shortest integer maximal or similar to γ, and $\mathbb{N}_0 = \{0,1,2,...\}$ where $\gamma \in \mathbb{N}$.

For extra details and characteristics of the FC, see [1,3].

3. Shifted Legendre polynomials

The well-known Legendre polynomials can be obtained by applying the next recurrence formula, which defines them on the interval $[-1,1]$, see [19]:

$$L_{h+1}(w) = \frac{2h+1}{h+1}\, w L_h(w) - \frac{h}{h+1} L_{h-1}(w), \quad h = 1,2,...,$$

$L_0(w) = 1$ and $L_1(w) = w$. Because of this polynomial on $y \in [0,1]$, we know the seeming polynomials shifted Legendre via inserting the revision of variable $w = 2y-1$. The polynomials shifted Legendre $L_k(2y-1)$ is read via $P_h(y)$. Then the $P_h(y)$ ability be gained as follows:

$$P_{h+1}(y) = \frac{(2h+1)(2y-1)}{(h+1)} P_h(y) - \frac{h}{h+1} P_{h-1}(y), \quad h = 1,2,..., \tag{4}$$

where $P_0(y) = 1$ and $P_1(y) = 2y-1$. The analytic shape of the polynomials shifted Legendre $P_h(y)$ of class h assumed via:

$$P_h(y) = \sum_{j=0}^{h} (-1)^{h+j} \frac{(h+j)!\, y^j}{(h-j)(j!)^2}. \tag{5}$$

Recall that $P_h(0) = (-1)^h$ and $P_h(1) = 1$. The orthogonality condition is:

$$\int_0^1 P_{j_1}(y) P_{j_2}(y)\, dy = \begin{cases} \dfrac{1}{2j_1+1}, & \text{for} \quad j_1 = j_2; \\ 0, & \text{for} \quad j_1 \neq j_2. \end{cases} \tag{6}$$

The function $x(y)$, square-integrable in $[0,1]$, must be considered in terms of polynomials shifted Legendre as:

$$x(y) = \sum_{j=0}^{\infty} x_j P_j(y),$$

where the coefficients x_j are presented by:

$$x_j = (2j+1) \int_0^1 x(z) P_j(y)\, dy, \quad j = 1,2,\ldots.$$

Only the first $(r+1)$-terms shifted Legendre polynomials have reasoned in the workout. Next, we have the following:

$$x_r(y) = \sum_{j=0}^{r} x_j P_j(y). \tag{7}$$

4. Steps the solution using the presented method

This section discusses the numerical solution of the FIDEs (1) using the least squares approach and shifted Legendre polynomials as a tool. The style is founded on the solution of the obscure functions $z_j(y)$ as

$$z_k(y) = \sum_{j=0}^{r} a_j^i P_j(x), \quad 0 \leq y \leq 1, \tag{8}$$

where, $P_j(y)$ is shifted Legendre polynomial and $A_j,\ j = 0,1,2,...$ are parameters. With compensation (8) in (1), as follows

$$D^\gamma \sum_{j=0}^{r} A_j P_j(y) = g_j(y) + \int_0^1 k_j(y,t) \left(\sum_{k=1}^{r} \gamma_{jk} \left[\sum_{j=0}^{r} A_j P_j(t) \right] \right),\ dt \tag{9}$$

The residual equation is known as

$$R_i(x,A_0,A_a,...,A_r) = \sum_{j=0}^{r} A_j D^\gamma P_j(y) - \int_0^1 k_j(y,t) \left(\sum_{k=1}^{r} \gamma_{jk} \left[\sum_{j=0}^{r} A_j P_j(t) \right] \right) dt - g_j(y). \tag{10}$$

Let

$$S_i(A_0,A_a,...,A_r) = \int_0^1 \left[R_1(y,A_0,A_a,...,A_r)\right]^2 w(y)\, dy, \tag{11}$$

anywhere, $w(y)$ is the weight positive function determined on $[0,1]$, in this mission we pick $w(y) = 1$.

$$S_i(A_0,A_a,...,A_r) = \int_0^1 \left\{ \sum_{j=0}^{r} A_j D^\gamma P_j(y) - \int_0^1 k_j(y,t) \left(\sum_{k=1}^{r} \gamma_{jk} \left[\sum_{j=0}^{r} A_j P_j(t) \right] \right) dt - g_j(y) \right\} dy. \tag{12}$$

Consequently, determining $A_j,\ j = 0,...,r$ that minimize S_i is equal to feedback on the better approach for the answer of the LFIDES (1).

Through tuning, the S_i value minimum is obtained.

$$\frac{\partial S_i}{\partial A_j} = 0, \quad j = 0,1,...,r, \tag{13}$$

$$\begin{aligned}&\int_0^1 \left\{ \sum_{j=0}^{r} A_j D^\gamma P_j(x) - \int_0^1 k_j(x,t) \left[\sum_{k=1}^{r} \gamma_{jk} \sum_{j=0}^{r} A_j P_j(t) \right] dt - g_j(y) \right\} \times \\ &\left\{ D^\gamma P_j(x) - \int_0^1 k_j(x,t) \left[\sum_{k=1}^{r} \gamma_{jk} \sum_{j=0}^{r} A_j P_j(t) \right] dt \right\} dy.\end{aligned} \tag{14}$$

We can generate a system of $(r+1)$ linear equations with $(r+1)$ unknown coefficients A_j by evaluating the above equation for $j = 0,...,r$. Matrices form can be used to build this system in the following way:

$$A = \begin{pmatrix} \int_0^1 R_i(x,A_0)h_0^i dx & \int_0^1 R_i(x,A_1)h_0^i dx & \cdots & \int_0^1 R_i(x,A_r)h_0^i dx \\ \int_0^1 R_i(x,A_0)h_1^i dx & \int_0^1 R_i(x,A_1)h_1^i dx & \cdots & \int_0^1 R_i(x,A_r)h_1^i dx \\ \vdots & \vdots & \vdots & \vdots \\ \int_0^1 R_i(x,A_0)h_n^i dx & \int_0^1 R_i(x,A_1)h_n^i dx & \cdots & \int_0^1 R_i(x,A_r)h_n^i dx \end{pmatrix}, \tag{15}$$

$$B = \begin{pmatrix} \int_0^1 g_j(y)h_0^i dx \\ \int_0^1 g_j(y)h_1^i dx \\ \vdots \\ \int_0^1 g_j(y)h_n^i dx \end{pmatrix}, \tag{16}$$

where

$$R_i(y,A_j) = \sum_{j=0}^{r} A_j D^\gamma P_j(x) - \int_0^1 k_j(y,t) \left[\sum_{k=1}^{n} \gamma_{jk} \sum_{j=0}^{r} A_j P_j(t) \right] dt, \tag{17}$$

$$h_j^i = D^\gamma P_j(y) - \int_0^1 k_j(y,t) \left[\sum_{k=1}^{n} \gamma_{jk} \sum_{j=0}^{n} A_j P_j(t) \right] dt \ j = 0,...,r, \ i = 1,...,r. \tag{18}$$

We can determine the values of the unknown coefficients and the approximate outcome of (1) by solving the system described above.

5. Simulations technique

Several numerical examples of FIDEs are provided in this section to support the findings above. All outcomes are produced by using the Mathematica 12 software.

Example 1. Let

$$\left.\begin{aligned} D^{\frac{2}{3}}y_1(z) &= \frac{-z}{6}+\frac{3z^{\frac{1}{3}}}{\Gamma(\frac{1}{3})}+\int_0^1 2zt[y_1(t)+y_2(t)]\,dt,\\ D^{\frac{2}{3}}y_2(z) &= \frac{5z^3}{6}+\frac{9z^{\frac{4}{3}}}{2\Gamma(\frac{1}{3})}+\int_0^1 z^3[y_1(t)-y_2(t)]\,dt \end{aligned}\right\}, \tag{19}$$

with the conditions $y_1(0)=-1$, $y_2(0)=0$, the accurate solution $y_1(z)=z-1$ and $y_2(z)=z^2$.

By employing an algorithm known as least squares and shifting the Legendre polynomials in collocation. With the obscure coefficients A_j, $j=0,\dots,4$, $i=1,\dots,r$, and (16), we have a system of (15) linear equations. The results using SLM are according to the accurate solution (see Fig. 1, Tables 1 and 2) and the paper of Saleh et al. [30].

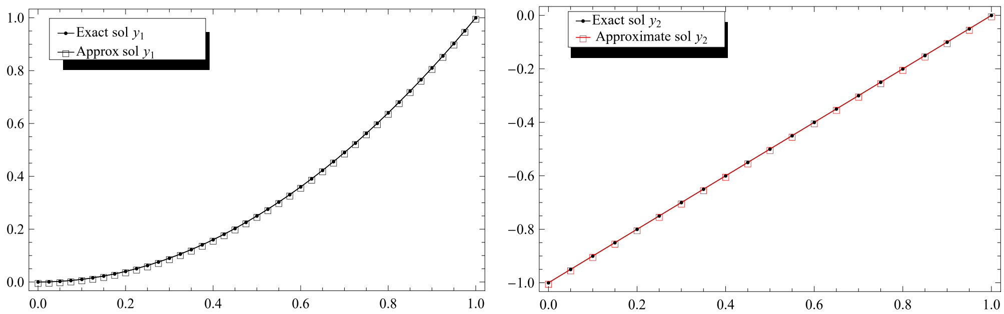


Fig. 1. The relation between the numerical solution and accurate solution

Table 1. Results for Example 1 at $n=4$ for $y_1(z)$, compare the accurate solution, the close solution, and the error

x	Accurate solution $y_1(z)$	Approximate solution $y_1(z)$	Error
0	–1	–1	0
0.2	–0.8	–0.798878	$1.122e-3$
0.4	–0.6	–0.598868	$1.132e-3$
0.6	–0.4	–0.399876	$1.24e-4$
0.8	–0.2	–0.198896	$1.104e-3$
1	0	0	0.0002125

Table 2. Results for Example 1 at $n=4$ for $y_2(z)$, compare the accurate solution, the close solution, and the error

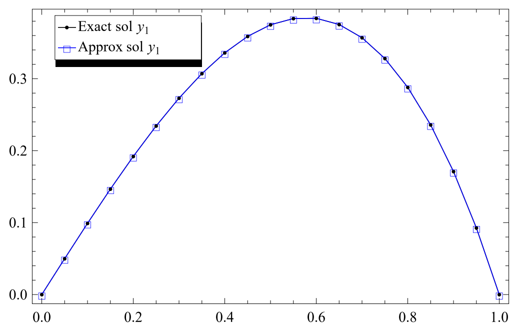
x	Accurate solution $y_2(z)$	Approximate solution $y_2(z)$	Error
0	0	0	0
0.2	0.04	0.03997987	$2.013e-5$
0.4	0.16	0.15999788	$2.12e-6$
0.6	0.36	0.35999677	$3.23e-6$
0.8	0.64	0.63999542	$4.58e-6$
1	1	0.9999987	$1.3e-6$

Example 2. Let

$$\left.\begin{aligned} D^{\frac{3}{4}}y_1(z) &= -\frac{1}{20}-\frac{z}{12}+\frac{4z^{\frac{1}{4}}(15-23z^2)}{15\Gamma(\frac{1}{4})}+\int_0^1 (z+t)[y_1(t)+y_2(t)],\, dt \\ D^{\frac{3}{4}}y_2(z) &= \frac{5z^3}{6}+\frac{9z^{\frac{4}{3}}}{2\Gamma(\frac{1}{3})}+\int_0^1 \sqrt{z}t^2[y_1(t)-y_2(t)]\, dt \end{aligned}\right\}. \tag{20}$$

Subject to $y_1(0)=0$, $y_2(0)=0$ with accurate solution $y_1(z)=z-z^3$, $y_2(z)=z^2-z$.

The same in the above example, using the style of least squares for the help of shifted Legendre polynomials collocation $P_j(z)$, $j=0,1,\ldots,r$ at $r=4$ to (20), the numerical results have displayed in Figure 2. We are compared from accurate solution and the approximation solution. The results using SLM are according to the accurate solution (see Tables 3 and 4) and the paper of Saleh et al. [30].



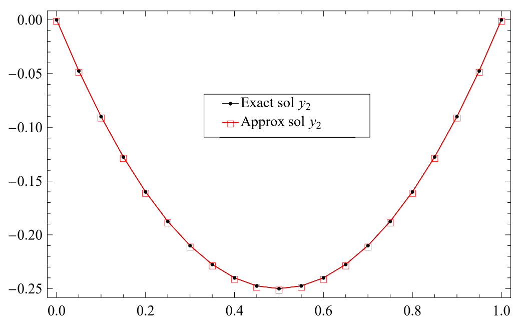


Fig. 2. The relation between the numerical solution and exact solution

Example 3. An equation with fractional integral-differential being studied:

$$D^{0.75}y(z)=\frac{z^{0.25}}{\Gamma(1.25)}-z^2-\frac{z^4}{3}+z\,y(z)+\int_0^z z\,s\,y(s)\,ds,$$

$$y(0)=0, \tag{21}$$

with the accurate solution $y(z)=z$.

Table 3. Results for Example 2 at $n=4$ for $y_1(z)$, compare the accurate solution, the close solution, and the error

x	Accurate solution $y_1(z)$	Approximate solution $y_1(z)$	Error
0	0	$-5.24553e-13$	$5.24553e-11$
0.2	0.192	0.192	$5.39485e-11$
0.4	0.336	0.336	$5.4956e-11$
0.6	0.384	0.384	$5.55445e-11$
0.8	0.288	0.288	$5.57221e-13$
1	0	$-5.57887e-13$	$5.57887e-13$

Table 4. Results for Example 2 at $n=4$ for $y_2(z)$, compare the accurate solution, the close solution, and the error

x	Accurate solution $y_2(z)$	Approximate solution $y_2(z)$	Error
0	0	$1.41276e-13$	$1.41276e-13$
0.2	–0.16	–0.192	$1.71613e-11$
0.4	–0.24	–0.24	$1.94622e-11$
0.6	–0.24	–0.24	$2.10276e-12$
0.8	–0.16	–0.16	$2.18492e-13$
1	0	$2.1438e-13$	$2.1438e-13$

Table 5. The outcomes of Example 3

z	Accurate	SLM	Abs. E
0.2	0.2	0.1999984476	1.5524E-6
0.4	0.4	0.3998762025	1.237975E-4
0.6	0.6	0.5983168111	1.6831889E-3
0.8	0.8	0.7887498311	1.12501689E-2
1.0	1.0	0.9486188796	5.13811204E-2

The results using SLM are according to the accurate solution (see Table 5) and the paper of Saleh et al. [25].

Example 4. An equation with fractional integral-differential is studied:

$$D^{0.25}y(z) = \frac{6\, z^{2.75}}{\Gamma(3.75)} - \frac{1}{5}\, z^2\, e^z\, y(z) + \int_0^z e^z\, s\, y(s)\, ds,$$

$$y(0) = 0, \tag{22}$$

with the accurate solution $y(z) = z^3$.

The results using SLM are according to the accurate solution (see Table 6) and the paper of Saleh et al. [25].

Table 6. The outcomes of Example 4

z	Accurate	SLM	Abs. E
0.2	0.008	0.00800000	0
0.4	0.064	0.06400000	0
0.6	0.216	0.21600000	0
0.8	0.512	0.51200000	0
1.0	1.000	1.00000000	0

6. Conclusion

In this essay, we have discussed the numerical style for resolving the FIDEs. We have used the least squares method and the Legendre method. We have used the Caputo meaning and have shown the properties of the FC. Additionally, we have shown the properties of the Legendre and shifted Legendre style and we have discussed the four examples. The outcomes offer that styles collocate for the numbering of terms is maximum. Because the solution is seen as a truncated shifted Legendre polynomials series, it is possible to easily estimate it given time spot values. Several numerical examples (four examples) have been given to clarify the notional outcomes and liken them to those gained using exact solutions. Figures and tables which obtain the view show the simulations of the suggested approaches that were used to get the theoretical results. Using Mathematica 12, we have programmed the numerical results.

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