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A NUMERICAL METHOD FOR SOLVING A CLASS OF VARIABLE-ORDER DIFFERENTIAL EQUATIONS USING HOSOYA POLYNOMIAL OF SIMPLE PATHS

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Abstract. In this paper, we use the operational matrices (OMs) and collocation method (CM) to obtain a numerical solution for a class of variable-order differential equations (VO-DEs). The fractional derivatives and the VO-derivatives are in the Caputo sense. The operational matrices are computed based on the Hosoya polynomials (HPs) of simple paths. Firstly, we assume the unknown function as a finite series by using the Hosoya polynomials as the basis functions. To obtain the unknown coefficients of this approximation, we computed the operational matrices of all terms of the main equations. Then, by using the operational matrix and collocation points, the governing equations are converted to a set of algebraic equations. Finally, an approximate solution is obtained by solving those algebraic equations.

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1. Introduction

Fractional calculus, which denotes differentiation and integration with fractional orders, has a history spanning more than three centuries. Many real-world phenomena can be more accurately described using fractional operators rather than ordinary calculus [1]. Fractional calculus has been identified as a very efficient mathematical tool in many scientific and engineering disciplines, such as complicated dynamic systems, phenomena, or structures [2]. Fractional calculus has been used to represent a wide range of real-world phenomena, including the transport of heat, finance, geohydrology, and medicine [3–7].

Systems may change depending on the time, place, or other conditions [8]. In 1993, Ross and Samko presented the concept of variable-order (VO) fractional operators and evaluated their features [9]. To precisely describe complicated physical systems and procedures, VO fractional calculus has attracted as a mathematical structure [8]. Since then, numerous researchers have generalized the VO fractional calculus, and its applications have been researched in numerous fields, such as petroleum engineering, viscoelasticity oscillators, engineering, signal processing, etc. [10].

Because the VO operators have a variable exponent kernel, the analysis of variable--order systems is more complicated than that of constant-order systems [11]. Obtaining analytical solutions is frequently challenging or even impossible. Therefore, numerous scholars are investigating numerical methods [10].

The finite difference approach is used to solve the VO-differential equations (VO-DEs) in [12]. To solve the cable VO-differential equation, Bhrawey and Zaky [13] used shifted Legendre polynomials. Tavares et al. [14] solved the VO partial differential equations by using a numerical approximation. The authors of [15] solved the VO problems using the kernel approach and boundary conditions. Refice et al. [16] have converted nonlinear fractional integro-differential equations of VO with multiterm boundary value conditions into ordinary Caputo's fractional differential equations of constant order.

In [17], a type of VO diffusion-wave equations have been solved by using operational matrices based on Bernstein polynomials. Jafari et al. solved a class of VO stochastic advection diffusion equations by using the collocation method (CM) and the operational matrix based on Hosoya polynomials of simple paths [18]. A class of VO-differential equations have been solved via the Ritz-approximation method and Genocchi polynomials in [10]. In [19], a numerical method for solving VO-fractional differential equations is used to research the dynamics of a circulant Halvorsen system.

Variable-order fractional differential equations (VO-FDEs) can be significantly simplified using polynomials, which can be transformed into an algebraic system of equations. In [17, 18, 20–24], the authors employed OMs based on various polynomials for solving various VO problems.

In this study, we consider the following type of variable-order differential equations

$$\begin{cases} {}^{C}\mathcal{D}_t^{\beta_1(t)}x(t) = f(t, x(t), {}^{C}\mathcal{D}_t^{\beta_2(t)}x(t), \dots, {}^{C}\mathcal{D}_t^{\beta_n(t)}x(t)), \quad \underbrace{k_i - 1 < \beta_i(t) \le k_i}_{\{k_i \in \mathbb{Z}^+\}_{i=1}^{n\in\mathbb{N}}}, \quad 0 \le t \le 1, \\ x^{(i)}(0) = a_i \in \mathbb{R}, \qquad i = 0, 1, \dots, k_1 - 1, \end{cases} \tag{1}$$

where $x(t)$ is the unknown function and ${}^{C}\mathcal{D}_t^{\beta_i(t)}$, with condition $\beta_1(t) > \beta_2(t) > \dots > \beta_n(t)$ is the Caputo derivative of VOs, which is defined as follows [25]

$${}^{C}\mathcal{D}_t^{\beta_i(t)}x(t) = \frac{1}{\Gamma(k_i - \beta_i(t))} \int_0^t (t-s)^{k_i - \beta_i(t) - 1} x^{(k_i)}(s)\, ds,$$

for the special case, when $x(t) = t^{\alpha}$, we have

$$\begin{cases} {}^{C}\mathcal{D}_t^{\beta(t)} t^{\alpha} = \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta(t)+1)} t^{\alpha-\beta(t)}, & \alpha \in \mathbb{N} \text{ and } \alpha \geq \lceil \beta(t) \rceil \\ & \text{or } \alpha \notin \mathbb{N} \text{ and } \alpha > \lfloor \beta(t) \rfloor \\ 0, & \alpha \in \mathbb{Z}^{+} \text{ and } \alpha < \lceil \beta(t) \rceil \end{cases} \tag{2}$$

The aim of this study is to convert problem (1) into an algebraic system of equations. For this purpose, we use operational matrices and the collocation method. In other words, (1) is transformed into several dependent matrices with the help of the operational matrices, which is converted to an algebraic system of equations by using the collocation points. The computational work is considerably reduced by operational matrices based on Hosoya polynomials.

This article has been structured as follows: In Section 2, the Hosoya polynomials and operational matrix are expressed. In Section 3, the numerical approach is explained. In Section 4, a few examples are provided to demonstrate the method's efficacy. A conclusion is presented in Section 5.

2. OM for constant / variable-order derivatives based HPs

In this section, we will briefly recall how to obtain the HPs of simple paths and then we will describe computing the operational matrices of ordinary, fractional, as well as variable-order derivatives of a function by using the HPs of simple paths.

2.1. The HPs of simple paths

The Hosoya polynomial was presented by Hosoya in 1998. The Hosoya polynomials are calculated for a variety of graphs such as Benzenoid graphs, trees, Hanoi, Polypheneyl chains, Fibonacci and Lucas graphs [18].

Definition 1. Let graph G be connected, Hosoya polynomials are defined as follows [26]:

$$\mathscr{H}(G,\zeta) = \sum_{r\geq 0} d(G,r)\zeta^{r}, \tag{3}$$

where $d(G,r)$ is the number of pairs of vertices with distance r and parameter ζ. □

In view of (3), the Hosoya polynomial for a path graph P_i with i vertices, where $i = 1,2,\ldots,M+1$, can be obtained as follows:

$$
\begin{aligned}
\mathscr{H}(P_1,\zeta) &= \sum_{r=0}^{1} d(P_1,r)\zeta^r = 1,\\
\mathscr{H}(P_2,\zeta) &= \sum_{r=0}^{2} d(P_2,r)\zeta^r = 2+\zeta,\\
\mathscr{H}(P_3,\zeta) &= \sum_{r=0}^{3} d(P_3,r)\zeta^r = 3+2\zeta+\zeta^2,\\
&\vdots\\
\mathscr{H}(P_{M+1},\zeta) &= \sum_{r=0}^{M+1} d(P_{M+1},r)\zeta^r = (M+1)+(M)\zeta+(M-1)\zeta^2+\ldots+\zeta^M.
\end{aligned}
\tag{4}
$$

Obviously, above polynomials are independent and can be used as basic functions for approximation. Let $x(t) \in L^2[0,1]$, $x(t)$ can be approximated by the HPs as follows:

$$x(t) = \sum_{r=1}^{\infty} c_r \mathscr{H}_r(t). \tag{5}$$

By choosing $M+1$ terms from equation (5), $x(t)$ can be approximated as follows:

$$x(t) \simeq x_{M+1}(t) = \sum_{r=1}^{M+1} c_r \mathscr{H}_r(t) = C^T\, \mathfrak{H}(t), \tag{6}$$

with

$$\mathfrak{H}(t) = [\mathscr{H}_1(t), \mathscr{H}_2(t), \ldots, \mathscr{H}_{M+1}(t)]^T,$$

and $C = [c_1, c_2, \ldots, c_{M+1}]^T$, where c_r is determined as follows:

$$C = Q^{-1} \int_0^1 x(t)\ \mathfrak{H}_r(t)dt,$$

where

$$Q^{-1} = \int_0^1 \mathfrak{H}_r(t)\mathfrak{H}_r^T(t)dt.$$

2.2. OM matrices based on HPs

First, we will describe how to obtain operational matrices based on HPs that will be employed in our numerical method. To accomplish this, we define the basis vector $\mathfrak{H}(t)$ as follows:

$$\mathfrak{H}(t) = [\mathscr{H}_1(t), \mathscr{H}_2(t), \ldots, \mathscr{H}_{M+1}(t)]^T = A\, T_M(t), \tag{7}$$

where $A = [a_{r,s}], r,s = 1,...,M+1$, with

$$a_{r,s} = \begin{cases} r-s+1, & r \geq s, \\ 0, & r < s, \end{cases}$$

and

$$T_M(t) = [1,t,...,t^M]^T.$$

From equation (7), we will have

$$T_M(t) = A^{-1}\mathfrak{H}(t)$$

According to (7) and the first-time derivative, we have

$$\frac{d}{dt}\mathfrak{H}(t) = \frac{d}{dt}(AT_M(t)) = \mathscr{D}\,\mathfrak{H}(t),$$

Where D is the OM of the derivative based on the HPs and defined as follows:

$$\mathscr{D} = \begin{bmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \\ -2 & 2 & 0 & ... & 0 & 0 \\ -2 & -2 & 3 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -2 & -2 & -2 & ... & M & 0 \end{bmatrix}.$$

In addition, for the $n \geq 2$, we have

$$\frac{d^n}{dt^n}\mathfrak{H}(t) = \mathscr{D}^n\,\mathfrak{H}(t). \tag{8}$$

Theorem 1. *The operational matrix for α-order fractional derivative of the vector $\mathfrak{H}(t)$ provided by* (7) *can be estimated as follows:*

$$D^{\alpha}\mathfrak{H}(t) \simeq P\mathfrak{H}(t), \tag{9}$$

where P is called the OM of fractional derivative based on HPs of simple paths. □

PROOF

$$\begin{aligned} D^{\alpha}(\mathfrak{H}(t)) &= D^{\alpha}\left[\mathscr{H}_1(t),\mathscr{H}_2(t),...,\mathscr{H}_{M+1}(t)\right]^T \\ &= \left[D^{\alpha}\mathscr{H}_1(t),D^{\alpha}\mathscr{H}_2(t),...,D^{\alpha}\mathscr{H}_{M+1}(t)\right]^T \end{aligned} \tag{10}$$

According to Caputo's derivative (2)

$$\begin{aligned} D^{\alpha}(\mathscr{H}_r(t) &= D^{\alpha}\left(\sum_{s=1}^{r}(r-s+1)t^{s-1}\right) = \sum_{s=1}^{r}(r-s+1)D^{\alpha}t^{s-1} \\ &= \sum_{s=1}^{r}(r-s+1)\frac{\Gamma(s)}{\Gamma(s-\alpha)}t^{s-\alpha-1}, r = 1,2,...,M+1 \end{aligned} \tag{11}$$

We approximate $t^{s-\alpha-1}$ by employing $\mathfrak{H}(t)$,

$$t^{s-\alpha-1} = \sum_{j=1}^{M+1} b_{s,j}\mathfrak{H}(t),$$

so

$$b_{s,j} = Q^{-1}\int_0^1 t^{s-\alpha-1}\mathfrak{H}(t), \tag{12}$$

by substituting Eq. (12) in the Eq. (11), we can write

$$D^{\alpha}(\mathscr{H}_r(t)) = \sum_{j=1}^{M+1}\left(\sum_{s=\lceil\alpha\rceil}^{r}(r-s+1)\frac{\Gamma(s)}{\Gamma(s-\alpha)}b_{s,j}\right)\mathfrak{H}(t), \tag{13}$$

$$D^{\alpha}\left[\mathscr{H}_1(t),\mathscr{H}_2(t),\dots,\mathscr{H}_{M+1}(t)\right]^T \simeq P\mathfrak{H}(t). \tag{14}$$

Theorem 2. *The operational matrix for VO-derivative $\beta(t)$ of $\mathfrak{H}(t)$ can be estimated as follows:*

$${}^{C}\mathcal{D}_t^{\beta(t)}\mathfrak{H}(t) = \Delta^{\beta(t)}\mathfrak{H}(t), \tag{15}$$

where

$$\Delta^{\beta(t)} = A\Phi^{\beta(t)}A^{-1}, \tag{16}$$

and $\Delta^{\beta(t)}$ is called the OM of VO-derivative based on the HPs. □

PROOF By applying the Caputo operator, ${}^{C}\mathcal{D}_t^{\beta(t)}$ to equation (7), we will have

$${}^{C}\mathcal{D}_t^{\beta(t)}\mathfrak{H}(t) = {}^{C}\mathcal{D}_t^{\beta(t)}(AT_M(t)) = A({}^{C}\mathcal{D}_t^{\beta(t)}T_M(t)). \tag{17}$$

Based on the definition of the Caputo operator, ${}^{C}\mathcal{D}_t^{\beta(t)}$ on the function t^i (2), we have [25]

$${}^{C}\mathcal{D}_t^{\beta(t)}t^i = \begin{cases} \dfrac{\Gamma(i+1)}{\Gamma(i-\beta(t)+1)}t^{i-\beta(t)}, & i\in\mathbb{N} \text{ and } i\geq\lceil\beta(t)\rceil \text{ or } i\notin\mathbb{N} \text{ and } i>\lfloor\beta(t)\rfloor, \\ 0, & i\in\mathbb{N}\cup\{0\} \text{ and } i<\lceil\beta(t)\rceil. \end{cases} \tag{18}$$

We consider $q=\lceil\beta(t)\rceil$ and substitute equation (18) in (17), so we have

$$\begin{aligned} {}^{C}\mathcal{D}_t^{\beta(t)}\mathfrak{H}(t) &= A[0 \quad 0 \quad \dots \quad 0 \quad \frac{\Gamma(q+1)t^{q-\beta(t)}}{\Gamma(q+1-\beta(t))} \quad \cdots \quad \frac{\Gamma(M+1)t^{M-\beta(t)}}{\Gamma(M+1-\beta(t))}]^T \\ &= A\Phi^{\beta(t)}T_M(t), \end{aligned}$$

where

$$\Phi^{\beta(t)} = [\chi_t^{u,w}], \quad u,w = 0,1,2,...,M,$$

with

$$\chi_t^{u,w} = \begin{cases} \dfrac{\Gamma(u+1)}{\Gamma(u+1-\beta(t))} t^{-\beta(t)}, & u = w \geq p, \\ 0, & \text{otherwise.} \end{cases}$$

From (7), we have

$${}^{C}\!D_t^{\beta(t)} \mathfrak{H}(t) = A\Phi^{\beta(t)} A^{-1} \mathfrak{H}(t) = \Delta^{\beta(t)} \mathfrak{H}(t),$$

3. Numerical approach of solving VO-DEs

In this section, we present a numerical approach based on the operational matrices and the collocation method to estimate the solution of (1).

At first, we approximate $x(t)$ in the form of a matrix based on the HPs presented in Equation (6), and then replace it into Equation (1) as follows:

$$\begin{cases} C^T D^{\beta_1(t)} \mathfrak{H}(t) = f(t, C^T \mathfrak{H}(t), C^T D^{\beta_2(t)} \mathfrak{H}(t), ..., C^T D^{\beta_n(t)} \mathfrak{H}(t)), \\ C^T D^i \mathfrak{H}(0) = a_i, \qquad i = 0,1,...,k_1 - 1. \end{cases} \tag{19}$$

After that, in view of (8), (9) and (15), we obtain the operational matrices for each term of Equation (19). In other words

$$\begin{cases} C^T \Delta^{\beta_1(t)} \mathfrak{H}(t) - f(t, C^T \mathfrak{H}(t), C^T \Delta^{\beta_2(t)} \mathfrak{H}(t), ..., C^T \Delta^{\beta_n(t)} \mathfrak{H}(t)) = 0, \\ C^T \mathscr{D}^i \mathfrak{H}(0) - a_i = 0, \qquad i = 0,1,...,k_1 - 1. \end{cases} \tag{20}$$

Now, by using the collocation points $t_l = \dfrac{l}{M}$, $l = 1,2,...,M+1-k_1$ in Equation (20), it leads to a system of algebraic equations.

The unknown coefficients c_i can be calculated by solving a system of algebraic equations. For solving that system, we use *Mathematica* software. Finally, an approximate solution be achieved by substituting vector C into equation (6). For obtaining the error bounds, we refer readers to [8, 17, 24].

4. Test problems

In this section, a few examples are solved with the presented approach for showing its performance.

Example 1. Consider the following VO-problem

$$ {}^{C}\!D_t^{\beta(t)}x(t) = t^2x(t) + \frac{15\sqrt{\pi}t^{\frac{5}{2}-\beta(t)}}{8\Gamma(\frac{7}{2}-\beta(t))} - t^{\frac{9}{2}}, \quad \beta(t) = 2-\sin^2(t), $$

with $x(0) = 0$, $x'(0) = 0$, where $x(t) = t^{\frac{5}{2}}$ is the exact solution of this problem. The numerical outputs of the suggested approach are designed for various values of M in Figure 1. By increasing the number of basis functions, it is obvious that the numerical solution provided by the suggested approach converges to the exact solution.

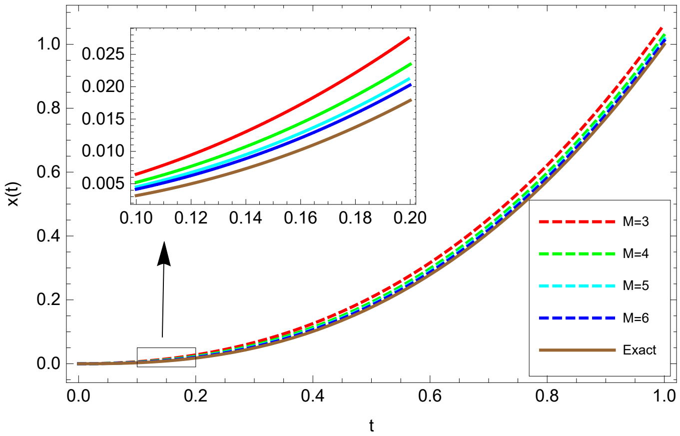

Fig. 1. The exact and approximate solutions given by various values of M for Example 1

Example 2. Consider the following nonlinear VO-problem

$$ {}^{C}\!D_t^{\beta(t)}x(t) = \sin(t)x^2(t) + \frac{e^t t^{0.5e^{-t}}H}{\Gamma(1+0.5e^{-t})} - e^{2t}\sin(t), \quad t \in [0,1], \quad \beta(t) = 1-0.5e^{-t}, $$
$$ x(0) = 1, $$

where $H = {}_1F_1[0.5e^{-t}, 1+0.5e^{-t}, -t]$ is the Kummer confluent hypergeometric function. We solved this problem, and the results have been presented in Figure 2, which shows the approximate solutions achieved by various values of M plus the exact solution $x(t) = e^t$. By increasing the number of basis functions, it is evident that the approximate solution converges to the exact solution. Table 1, displays the absolute error (AE) at many selected points with various values of M. □

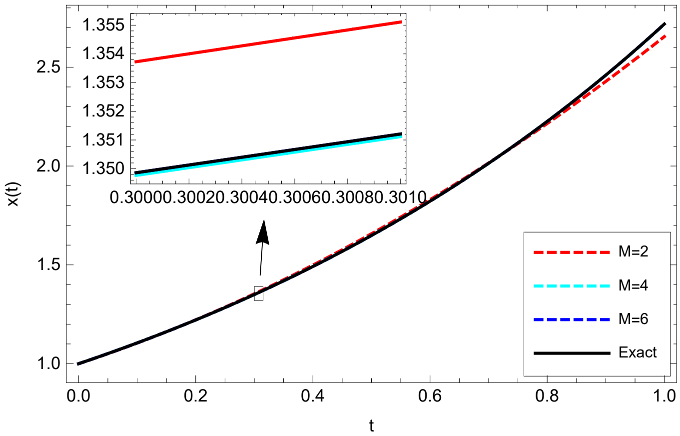

Fig. 2. The exact and approximate solutions for Example 2

Table 1. Comparison of the AE for Example 2

t	$M=2$	$M=4$	$M=6$
0.2	$8.2944e-4$	$1.0945e-4$	$3.6071e-7$
0.4	$6.9798e-3$	$8.8150e-5$	$3.5421e-7$
0.6	$7.5980e-3$	$1.4064e-4$	$5.5333e-7$
0.8	$1.0572e-2$	$3.0742e-4$	$1.2287e-6$

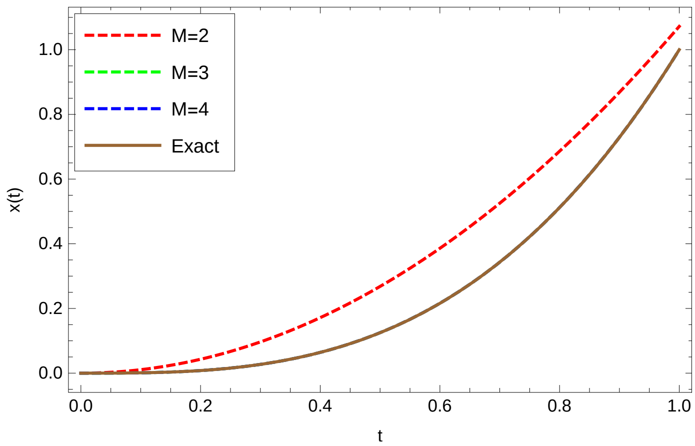

Fig. 3. The exact and approximate solutions for Example 3

Example 3. Consider the following multi VO-problem [20]

$$\begin{cases} {}^{C}\mathcal{D}_t^{\beta_1(t)}x(t) & = -\sin(t)\,{}^{C}\mathcal{D}_t^{\beta_2(t)}x(t) - \cos(t)x(t) + t^3\cos(t) + \\ & \dfrac{6\sin(t)t^{3-\beta_2(t)}}{\Gamma(4-\beta_2(t))} + \dfrac{6t^{3-\beta_1(t)}}{\Gamma(4-\beta_1(t))}, \quad t \in [0,1], \\ x(0) = 0, & x'(0) = 0, \end{cases}$$

where $\beta_1(t)$ and $\beta_2(t)$ are $2-\sin^2(t)$ and $1-\dfrac{e^{-t^3}}{6}$, respectively. Using the suggested approach, the numerical results along with the exact solution $x(t)=t^3$ are indicated in Figure 3. As illustrated in Figure 3, increasing the number of basis functions causes the numerical solution to converge to the exact solution. The suggested approach offers results that are nearly identical to the methods presented in [20]. Table 2 shows the AE at some selected points with differing values of $\beta_1(t)$ and $\beta_2(t)$ when $M=5$.

Table 2. Comparison of the AE for Example 3

t	$\beta_1(t),\beta_2(t)$		
	$1+\sin^2(t),1-\sin^2(t)$	$2-\frac{t}{2},\frac{t}{2}$	$2-0.5e^{-t},1-0.5e^{-t}$
0.2	$3.5475e-15$	$5.3239e-15$	$4.4357e-15$
0.4	$6.1756e-15$	$1.0617e-14$	$8.8402e-15$
0.6	$6.1062e-15$	$1.4988e-14$	$1.2324e-14$
0.8	$4.9960e-15$	$1.5654e-14$	$1.5654e-14$

Example 4. As the last example, consider [27,29]

$$
\begin{aligned}
&{}^{C}\mathcal{D}_t^{\beta(t)}x(t)=\frac{2t^{2-\beta(t)}}{\Gamma(3-\beta(t))}+\frac{3t^{1-\beta(t)}}{\Gamma(2-\beta(t))},\quad t\in[0,1],\quad \beta(t)=\sin(t)\text{ and }\frac{t}{2},\\
&x(0)=0,
\end{aligned}
$$

The exact solution is $x(t)=t^2+3t$. Putting $M=2$ and utilizing the presented approach, we obtain the unknown coefficients as follows:

$$c_1=-5,\quad c_2=1,\quad c_3=1,$$

this provides the exact solution. Based on the results of [27, 29], our approach provides the exact solution with only a minimal number of basis functions ($M=2$). The authors of article [27] used the method presented in article [28] and obtained the greatest absolute error $E_\infty(M)$. The obtained numerical results from [27,29] and the method presented in article [28] are indicated in Table 3.

Table 3. Comparison of the $E_\infty(M)$ for other numerical methods for Example 4

M	$\beta(t)=\sin(t)$			$\beta(t)=\frac{t}{2}$		
	$E_\infty(M)$ [27]	$E_\infty(M)$ [28]	$E_\infty(M)$ [29]	$E_\infty(M)$ [27]	$E_\infty(M)$ [28]	$E_\infty(M)$ [29]
4	$2.47e-2$	$2.87e-1$	$7.53e-7$	$5.98e-3$	$2.12e-1$	$1.20e-6$
8	$5.60e-3$	$1.44e-1$	$3.16e-10$	$1.42e-3$	$1.06e-1$	$3.34e-9$
16	$1.33e-3$	$7.26e-2$	$2.11e-10$	$3.47e-4$	$5.30e-2$	$1.08e-10$

5. Conclusion

In this study, we presented a numerical approach for solving variable-order differential equations. We approximated all terms of the given VO-DEs by operational matrices based on the Hosoya polynomials of simple paths. After that, we used the collocation method to obtain numerical result. A few numerical examples have been solved, and the results are compared with the other numerical methods. The results attained showed the approach's superiority over other existing methods.
Mathematica has been used for computation.

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